

DIRICHLET FORMS AND CRITICAL EXPONENTS ON FRACTALS

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ABSTRACT. Let $B_{2,\infty}^\sigma$ denote the Besov space defined on a compact set $K \subset \mathbb{R}^d$ with an α -regular measure μ . The *critical exponent* σ^* is the largest σ such that $B_{2,\infty}^{\sigma^*}$ remains non-trivial. The exponent is determined by the geometry of K and μ . In the analysis of fractals, it is known that for many standard self-similar sets K , $B_{2,\infty}^{\sigma^*}$ is the domain of some local regular Dirichlet forms. In this paper, we study two anomalous p.c.f. fractals K . On the first K , we provide two constructions of the local regular Dirichlet forms that do not have $B_{2,\infty}^{\sigma^*}$ as domain; one satisfies the well-known energy self-similar identity, the other one does not, and is not a conventional kind. For the second K , we show that the associated Besov space has two critical exponents, which is different from the usual perception. In the proof, we first discretize the Besov norm in terms of the boundary of the p.c.f. set, then determine the critical exponents and construct the Dirichlet forms through some electrical network techniques.

1. Introduction

Let K be a closed subset in \mathbb{R}^d with the Euclidean metric, and let μ be an α -regular measure on K , that is, there exists $\alpha > 0$ such that for any ball $B(x, r)$ with $0 < r < \text{diam}(K)$,

$$\mu(B(x, r)) \asymp r^\alpha. \quad (1.1)$$

(Here $f \asymp g$ means there exists constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$.) Fix $\sigma > 0$, for $u \in L^2(K, \mu)$, let

$$[u]_{B_{2,\infty}^\sigma}^2 := \sup_{0 < r < 1} r^{-\alpha-2\sigma} \int_K \int_{B(x,r)} |u(x) - u(y)|^2 d\mu(y) d\mu(x), \quad (1.2)$$

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and define $B_{2,\infty}^\sigma := \{u \in L^2(K, \mu) : \|u\|_{B_{2,\infty}^\sigma} < \infty\}$ with norm $\|u\|_{B_{2,\infty}^\sigma} := \|u\|_2 + [u]_{B_{2,\infty}^\sigma}$. The space is a Banach space and belongs to the class of Besov spaces (cf. for example [10], [6]; note that in [10], this space is denoted by $\text{Lip}(\sigma, 2, \infty)$.)

Obviously, $B_{2,\infty}^\sigma \subset B_{2,\infty}^{\sigma'}$ if $0 < \sigma' < \sigma$. The space $B_{2,\infty}^\sigma$ can be dense in $C(K)$, or dense in $L^2(K, \mu)$; it can also become trivial as σ increases, depending on the geometry of K and μ . Let us define the *critical exponents* on (K, μ) by

$$\sigma^* := \sup \{\sigma : B_{2,\infty}^\sigma \cap C(K) \text{ is dense in } C(K)\},$$

For many self-similar sets K , $B_{2,\infty}^{\sigma^*}$ are the domains of some local regular Dirichlet forms (if exist), and they are essential in the study of the Laplacians, Brownian motions, and the associated heat kernels (e.g., [1, 6, 9, 10, 12, 17, 18]). The value $\beta^* = 2\sigma^*$ is called the *walk dimension* of (K, μ) . It is an important parameter in the study of the heat kernels, which corresponding to the speed of diffusion on the underlying sets. Heuristically, the larger the value β^* , the harder is for the diffusion process (Brownian motion) to drift away from the initial position. It is well-known that if K is a domain in \mathbb{R}^d , then $\beta^* = 2$; if K is the d -dimensional Sierpinski gasket, then $\beta^* = \log(d+3)/\log 2$ [10]. There are also extensions on the nested fractals [17], and approximate values of some other specific cases (e.g. Sierpinski carpet [1]); for Cantor-type sets, $\beta^* = \infty$ [13]. More generally, the notion of Besov space and critical exponents can be extended to metric measure spaces (K, d, μ) , where (K, d) is a locally compact, separable metric space, and μ is α -regular as before. It is known that for $2 \leq \beta^*$, then with a heat kernel assumption and a *chain condition* on (X, d) , we have $2 \leq \beta^* \leq \alpha + 1$ [6].

In the previous study of $B_{2,\infty}^{\sigma^*}$, one often assumes that the space admits a Brownian motion with a Gaussian or a sub-Gaussian heat kernel. In such cases, it is known that if $\sigma > \sigma^*$, then $B_{2,\infty}^\sigma$ consists of constant functions only. It is nature to ask if the *a priori* assumption of existence of such heat kernel is necessary. For this we define another critical exponent

$$\sigma^\# := \sup \{\sigma : B_{2,\infty}^\sigma \text{ contains non-constant functions}\},$$

It is clear that $\sigma^* \leq \sigma^\#$, and as we note above, it has been taken for granted that $\sigma^* = \sigma^\#$ in literature. In this note, we will construct two anomalous examples of self-similar sets. The first example satisfies $\sigma^* = \sigma^\#$ and $B_{2,\infty}^{\sigma^*}$ is contained in $C(K)$, but is not dense there, hence it does not support a regular Dirichlet form; on the other hand, we can construct two different local regular Dirichlet forms on K (with domains different from $B_{2,\infty}^{\sigma^*}$). The second example is such that $\sigma^* < \sigma^\#$.

In this study, we will concentrate on the class of post critically finite (p.c.f.) self-similar sets [12]. For this class, we can discretize the expression in (1.2) in terms of the “boundary” of the p.c.f. set, which provides a discrete energy form on K to approximate $[u]_{B_{2,\infty}^\sigma}^2$. This setup was first used by Jonnson in [10] for the Sierpinski gasket. For the more general situation, we see that a number of properties that are obvious in the Sierpinski gasket, will need to be justified here. Let $\{F_i\}_{i=1}^N$ be an iterated function system (IFS) of the form $F_i(x) = \rho(x - b_i) + b_i$ with $0 < \rho < 1$, $b_i \in \mathbb{R}^d$, and let K be the self-similar set. Assume that the IFS has the p.c.f. property, let V_0 be the boundary of K , $V_n = \bigcup_{i=1}^N F_i(V_{n-1})$. We prove (Theorem 3.4)

Theorem 1.1. *With the p.c.f. set K defined as above, then*

$$[u]_{B_{2,\infty}^\sigma}^2 \asymp \sup_{j \geq 0} \left\{ \rho^{-(2\sigma-\alpha)j} \sum_{x,y \in F_\omega(V_0); |\omega|=j} |u(x) - u(y)|^2 \right\}. \quad (1.3)$$

The first example is given in Section 4 (see Figure 3), it is modified from the Vicsek cross by adding two eyebolts on the cross to produce the irregularity, we call it the *Vicsek eyebolted cross*. It consists of 21 maps with contraction ratio $1/9$, and has four boundary points $V_0 = \{p_1, p_2, p_3, p_4\}$. By using some techniques and a generalized Δ -Y transform from the electrical network theory, we show that (Theorems 4.3, 4.4)

Theorem 1.2. *For the Vicsek eyebolted cross K in Figure 3, the critical exponents are*

$$\sigma^* = \sigma^\# = \frac{1}{2} \left(1 + \frac{\log 21}{\log 9} \right),$$

Moreover,

- (i) $B_{2,\infty}^{\sigma^*} (\subset C(K))$ is dense in $L^2(K, \mu)$, but not dense in $C(K)$;
- (ii) there are two kinds of local regular Dirichlet forms that can be constructed on K , one satisfies the energy self-similar identity [12], the other one does not.

As is seen from (i), we can not have a *regular* (sufficiently many continuous functions) energy form that has domain $B_{2,\infty}^{\sigma^*}$, which is defined by the uniform conductances (as in the sum of (1.3)). On the other hand, in (ii), we can use different conductances to obtain energy forms that yield local regular Dirichlet forms on K . The first construction gives an energy form that satisfies the *energy self-similar identity* [12], which also provides a non-trivial concrete example for the abstract proof of the existence of such energy form in [7] and [16]. For the second construction of the Dirichlet form, it is new up to the knowledge of the authors. It is worthwhile to note

that in order to balance out the asymmetry of K , the first construction is by setting the same conductance on each subcell, and putting “renormalizing factors” on the subcells; while the second construction does not depend on the renormalizing factor, but by varying the conductances in a different way.

The second example is given in Section 5 (see Figure 5), we call it a *Sierpinski sickle*. It is a connected p.c.f. set K generated by an IFS of 17 similitudes of contraction ratio $1/7$; the boundary has three points $V_0 = \{p_1, p_2, p_3\}$. By using the electrical network technique of Δ -Y transform, we show that the resulting resistances of V_n on V_0 are (Proposition 5.3)

$$R_n(p_1, p_2) \asymp R_n(p_2, p_3) \asymp \left(\frac{17}{2}\right)^n, \quad \text{and} \quad R_n(p_3, p_1) \asymp 7^n. \quad (1.4)$$

The two different values together with the exponents in (1.3) allow us to show that (Theorem 5.4),

Theorem 1.3. *For the Sierpinski sickle (Figure 5), we have*

$$\sigma^* = \frac{1}{2} \left(1 + \frac{\log 17}{\log 7} \right), \quad \sigma^\# = \frac{1}{2} \left(\frac{2 \log 17 - \log 2}{\log 7} \right).$$

Moreover, $B_{2,\infty}^{\sigma^*} (\subset C(K))$ is dense in $C(K)$, and $B_{2,\infty}^{\sigma^\#}$ is dense in $L^2(K, \mu)$ (but not dense in $C(K)$).

We do not know if $B_{2,\infty}^{\sigma^*}$ will support a local regular Dirichlet form. However, similar to the first example, we can construct a self-similar energy form on K by assigning certain renormalizing factors explicitly on the IFS. On the other hand the second construction of the energy form does not work in this example.

For the organization of the paper, in Section 2, we recall the definition of a Dirichlet form, and prove some basic facts that is needed to discretize the Besov norm, we also introduce the notion of resulting resistance and the Δ -Y transform. We prove Theorem 1.1 in Section 3. By using this and the resulting resistances, we construct the two examples and prove Theorems 1.2 and 1.3 in Section 4 and 5. Finally in Section 6, we give some remarks on the two examples, and another related Besov space $B_{2,2}^\sigma$ of the non-local Dirichlet forms.

2. Preliminaries

We recall the standard definition of Dirichlet form. Let (M, d) be a locally compact, separable metric space, and let ν be a Radon measure on M such that $\text{supp}(\nu) = M$; the triple (M, d, ν) is called a *metric measure space*. Let $C_0(M)$ denote the space of continuous functions with compact support.

Definition 2.1. *On (M, d, ν) , a Dirichlet form \mathcal{E} with domain \mathcal{F} is a symmetric bilinear form which is non-negative definite, closed, densely defined on $L^2(M, \nu)$, and satisfies the Markovian property: $u \in \mathcal{F} \Rightarrow \tilde{u} := (u \vee 0) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}[\tilde{u}] \leq \mathcal{E}[u]$. An energy form is $\mathcal{E}[u] := \mathcal{E}(u, u)$, $u \in \mathcal{F}$.*

A Dirichlet form is called regular if $\mathcal{F} \cap C_0(M)$ is dense in $C_0(M)$ with the supremum norm, and dense in \mathcal{F} with the $\mathcal{E}_1^{1/2}$ -norm. It is called local if $\mathcal{E}(u, v) = 0$ for $u, v \in \mathcal{F}$ having disjoint compact supports.

The importance of a local regular Dirichlet form is that it can induce a *Laplacian* on M . However it is a non-trivial matter to construct or to prove the existence of such form. In fact there are only limited classes of self-similar sets that are known to admit Laplacians, and it is still an open question to construct Laplacians for larger classes of fractal sets. The p.c.f. sets [12] defined in the following is one of the known classes that gives a rather satisfactory solution to this question.

Let $\{F_i\}_{i=1}^N$ be an IFS on \mathbb{R}^d such that

$$F_i(x) = \rho(x - b_i) + b_i, \quad 1 \leq i \leq N. \quad (2.1)$$

where $0 < \rho < 1$ and $b_i \in \mathbb{R}^d$. Let $K = \bigcup_{i=1}^N F_i(K)$ be the self-similar set, and let μ be the self-similar measure defined by $\mu = \frac{1}{N} \sum_{i=1}^N \mu \circ F_i^{-1}$. If the IFS satisfies the open set condition (OSC), i.e., there is a nonempty bounded open set O such that $F_i(O) \subset O$ and $F_i(O) \cap F_j(O) = \emptyset$ for $i \neq j$, then the Hausdorff dimension of K is $\dim_H(K) = \alpha = \frac{\log N}{|\log \rho|}$, and μ is the α -Hausdorff measure normalized on K , it is α -regular in the sense of (1.1).

We define the symbolic space as usual. Let $\Sigma = \{1, \dots, N\}$ be the alphabets, Σ^n the set of words of length n , and Σ^∞ the set of infinite words $\omega = \omega_1 \omega_2 \dots$, and let $\pi : \Sigma^\infty \rightarrow K$ be defined by $\{x\} = \{\pi(\omega)\} = \bigcap_{n \geq 1} K_{\omega_1 \dots \omega_n}$, a symbolic representation of $x \in K$ by ω .

Following Kigami [12], we define the *critical set* C and the *post-critical set* \mathcal{P} for K by

$$C = \pi^{-1}\left(\bigcup_{1 \leq i < j \leq N} (K_i \cap K_j)\right), \quad \mathcal{P} = \bigcup_{m \geq 1} \tau^m(C),$$

where $K_i = F_i(K)$, $\tau : \Sigma^\infty \rightarrow \Sigma^\infty$ is the left shift by one index. If \mathcal{P} is a finite set, we call $\{F_i\}_{i=1}^N$ a *post-critically finite* (p.c.f.) IFS, and K is a p.c.f. self-similar set. The *boundary* of K is defined to be $V_0 = \pi(\mathcal{P})$. (We always assume K is connected and $\#(V_0) \geq 2$ to avoid triviality). We also define

$$V_n = \bigcup_{i \in \{1, \dots, N\}} F_i(V_{n-1}), \quad V_* = \bigcup_{n \geq 1} V_n.$$

It is clear that K is the closure of V_* . We call $V_\omega := F_\omega(V_0)$ a *cell* of V_n for any $\omega \in \Sigma^n$, where $F_\omega = F_{\omega_1} \circ \dots \circ F_{\omega_n}$.

It is known that a p.c.f. IFS in (2.1) satisfies the open set condition [3] (More generally, this is true if the associate similar matrices A_i of F_i (instead of the ρ in (2.1)) are commensurable i.e., there exists A such that $A_i = A^{n_i}$; but it is not true without this assumption [19].) We will prove another stronger separation property of the IFS. We say that an IFS satisfies condition (H) if

(H) *there exists $C_0 > 0$ such that for any integer $m \geq 1$ and any two words ω and ω' with length m and $K_\omega \cap K_{\omega'} = \emptyset$, then $\text{dist}(K_\omega, K_{\omega'}) \geq C_0 \rho^m$.*

This property on the Sierpinski gasket is obvious and was used in [10]. It was also studied independently in [15] in another context, and there are non-p.c.f. examples that condition (H) do not hold even with the open set condition. We will need the following proposition in Section 3 for the discretization of the Besov norm in (1.2).

Proposition 2.2. *Suppose the IFS $\{F_i\}_{i=1}^N$ in (2.1) has the p.c.f. property, then it satisfies condition (H).*

We first prove a lemma. We quote from [3, Proposition 2.4] that under the p.c.f. assumption, if $p \in K_i \cap K_j$, then p has two representations $i\theta_1\dot{\xi}, j\theta_2\dot{\eta} \in \Sigma^\infty$, i.e.,

$$\pi(i\theta_1\dot{\xi}) = \pi(j\theta_2\dot{\eta}) = p, \quad (2.2)$$

where $\dot{\xi}, \dot{\eta}$ are recurrent words; we can also choose θ_1 and θ_2 to be the shortest words respectively so that the above holds.

Lemma 2.3. *Suppose the IFS $\{F_i\}_{i=1}^N$ has the p.c.f. property, and $p \in K_i \cap K_j, i \neq j$. Then*

(i) *there exists k_0 (independent of i, j) such that for $|\tau| = |\tau'| > k_0$, $K_{i\tau} \cap K_{j\tau'}$ contains at most one point.*

(ii) *suppose ω, ω' are such that $K_\omega \cap K_{\omega'} = \emptyset$, and*

$$\omega = i\theta_1 \underbrace{\xi \cdots \xi}_{n_2 k} \xi' := i\theta_1 \xi^{n_2 k} \xi', \quad \omega' = j\theta_2 \underbrace{\eta \cdots \eta}_{n_1 k} \eta' := j\theta_2 \eta^{n_1 k} \eta'$$

where ξ, η are as in (2.2) with $|\xi| = n_1$, $|\eta| = n_2$, and $|\xi'|, |\eta'| \leq n_1 n_2$. Then

$$\text{dist}(K_\omega, K_{\omega'}) = \rho^{n_1 n_2 k} \text{dist}(K_{i\theta_1 \xi'}, K_{j\theta_2 \eta'}). \quad (2.3)$$

Proof. The p.c.f. assumption implies that the intersection $K_i \cap K_j$ is a finite set for any $i \neq j$. Let $k_{i,j}$ be sufficiently large such that for any $\theta, \theta' \in \Sigma^*$ with $|\theta|, |\theta'| \geq k_{i,j}$, $K_{i\theta} \cap K_{j\theta'}$ contains at most one point. Since there are only finitely many such pair i, j , we take $k_0 = \max_{i,j} \{k_{i,j}\}$ and it will satisfy the assertion in (i).

To prove (ii), we let $\bar{p}_\xi = \lim_{n \rightarrow \infty} K_{\xi^n}$. Then \bar{p}_ξ is a fixed point of F_ξ , and by observing that $|\xi| = n_1$, we have for any $n \geq 1$, $F_{\xi^n}(x) = \rho^{n_1 n}(x - \bar{p}_\xi) + \bar{p}_\xi$. Let $\bar{p}_{i\theta_1}$ be the fixed point of $F_{i\theta_1}$, then by (2.2), we have

$$p = \rho^{|\theta_1|+1}(\bar{p}_\xi - \bar{p}_{i\theta_1}) + \bar{p}_{i\theta_1}.$$

It follows that

$$\begin{aligned} K_{i\theta_1 \xi^{n_2 k} \xi'} - p &= \rho^{|\theta_1|+1}(K_{\xi^{n_2 k} \xi'} - \bar{p}_{i\theta_1}) + \bar{p}_{i\theta_1} - p \\ &= \rho^{|\theta_1|+1}(K_{\xi^{n_2 k} \xi'} - \bar{p}_\xi) \\ &= \rho^{|\theta_1|+1+n_1 n_2 k}(K_{\xi'} - \bar{p}_\xi) \\ &= \rho^{n_1 n_2 k}(K_{i\theta_1 \xi'} - p) \end{aligned}$$

Similarly, we have $K_{j\theta_2 \eta^{n_1 k} \eta'} - p = \rho^{n_1 n_2 k}(K_{j\theta_2 \eta'} - p)$. This implies (2.3). \square

Proof of Proposition 2.2 Note that the number of triples (i, j, p) in Lemma 2.3 is finite, and for each (i, j, p) , the total number of words $i\theta_1 \xi', j\theta_2 \eta'$ are finite. By (2.3), there exists $C_0 > 0$, such that for $|\omega| = |\omega'| = m$ with the form as in Lemma 2.3(ii),

$$\text{dist}(K_\omega, K_{\omega'}) \geq C_0 \rho^m. \quad (2.4)$$

Let k_0 be as in Lemma 2.3(i). Consider any two finite words ω and ω' of length $m \geq 2k_0$, and $K_\omega \cap K_{\omega'} = \emptyset$, we let $\omega|_n$ and $\omega'|_n$ denote the initial segments of ω and ω' of length n . The proposition follows from the following two cases.

(i) if $K_{\omega|_{k_0+1}} \cap K_{\omega'|_{k_0+1}} = \emptyset$, then we simply have

$$\text{dist}(K_\omega, K_{\omega'}) \geq \text{dist}(K_{\omega|_{k_0+1}}, K_{\omega'|_{k_0+1}}) \geq C_1 \geq C_1 \rho^m$$

where $C_1 > 0$ is the minimum of the distances of disjoint cells with word length $\leq k_0 + 1$.

(ii) Otherwise, let n_0 be the largest n with $k_0 < n < m$ and $K_{\omega|_n} \cap K_{\omega'|_n} \neq \emptyset$. Then,

$$K_{\omega|_{n_0+1}} \cap K_{\omega'|_{n_0+1}} = \emptyset.$$

As $n_0 > k_0$, by Lemma 2.3(i), $K_{\omega|_n} \cap K_{\omega'|_n}$ has exactly one intersection point, and $K_{\omega|_{n_0+1}}$ and $K_{\omega'|_{n_0+1}}$ has the form as in Lemma 2.3. By (2.4), we have

$$\text{dist}(K_\omega, K_{\omega'}) \geq \text{dist}(K_{\omega|_{n_0+1}}, K_{\omega'|_{n_0+1}}) \geq C_0 \rho^m.$$

□

For a function $u \in \ell(V_*)$, we can define an energy $\mathcal{E}_n[u]$ on the graph V_n , $n \geq 0$, by

$$\mathcal{E}_n[u] = \sum_{x,y \in F_\omega(V_0), |x|=|y|=n} c_n(x,y) |u(x) - u(y)|^2, \quad (2.5)$$

where $c_n(x,y)$ is the conductance of the nodes x,y . In literature, the most studied approach to construct a Laplacian on a p.c.f. self-similar set is to consider the sequence $\mathcal{E}_{n+1}[u] = \sum_{i=1}^N \tau_i^{-1} \mathcal{E}_n[u \circ F_i]$, where $0 < \tau_i < 1$ are the renormalizing factors. If the $\mathcal{E} = \lim_{n \rightarrow \infty} \mathcal{E}_n$ exists, then \mathcal{E} satisfies the *energy self-similar identity* (also called *regular harmonic structure*)

$$\mathcal{E}[u] = \sum_{i=1}^N \tau_i^{-1} \mathcal{E}[u \circ F_i], \quad u \in \mathcal{F}, \quad (2.6)$$

and defines a local regular Dirichlet form ([12], [18]). For the Sierpinski gasket, we can take $\tau_i = 3/5$ and $c_n(x,y) = (\frac{5}{3})^n$. In this case, the Dirichlet form \mathcal{E} on the metric measure space $(K, |\cdot|, \mu)$ (μ is the normalized α -Hausdorff measure on K) has domain $\mathcal{F} = B_{2,\infty}^{\sigma^*}$ with $\sigma^* = \frac{\log 5}{2 \log 2}$. A similar result is for the nested fractals [17]. More generally, for the τ_i are not all equal, (2.6) induces a Dirichlet form with domain on an analogous Besov space on the metric measure space (K, d_r, ν) , where d_r is the resistance metric on K , and ν is the self-similar measure with weights $\{\tau_i^s\}_{i=1}^N$ where $\sum_{i=1}^N \tau_i^s = 1$, and the domain \mathcal{F} is a modified Besov space with respect to (K, d_r, ν) ([12], [6], [9]).

We can consider (V_n, r_n) as an electrical network, with $r_n(x,y) = c_n(x,y)^{-1}$, $x,y \in V_n$ as resistance.

Definition 2.4. Two networks (V_n, r_n) and (V_m, r_m) , $n > m$, are said to be compatible if for any $u \in \ell(V_m)$,

$$\min \{ \mathcal{E}_n[v] : v \in \ell(V_n), v|_{V_m} = u \} = \sum_{x,y \in V_m} \frac{1}{r_m(x,y)} |u(x) - u(y)|^2.$$

We also call $r_m(x,y)$, $x,y \in V_m$ the resulting resistance of V_n on V_m . In particular for $m = 0$, we will use the notation $R_n(p,q)$, $p,q \in V_0$ to replace $r_0(p,q)$.

For the energy form in (2.5), we call a function h on V_n harmonic on a subset $E \subset V_n$ if $h(x) = \sum_{x \sim y} c_n(x, y)h(y)$, $x \in E$. In Definition 2.4, the function $v \in \ell(V_n)$ that attains the minimum is a harmonic function on $V_n \setminus V_m$ (always exists); we call it a *harmonic extension* of $u \in \ell(V_m)$ to V_n (e.g., [14, Proposition 4.1]). As v is harmonic on the “interior” of each subcell of V_m , we see that v is a “piecewise harmonic” function on V_n . In the sequel, we will use this method to extend a function $u \in \ell(V_m)$ inductively to all $n > m$, and consider the limiting function.

To evaluate the resulting resistance and estimate the energy functional on a graph, we will use some elementary techniques like the series law and parallel law of resistance and the Δ -Y transform. Recall the Δ -Y transform (see e.g., [12], [18]) states that the Δ -shaped resistors (R_{12}, R_{23}, R_{31}) and the Y-shaped resistors (a, b, c) in Figure 1 in any network are equivalent by the following relation

$$a = \frac{R_{12}R_{31}}{R}, \quad b = \frac{R_{12}R_{23}}{R}, \quad c = \frac{R_{31}R_{23}}{R}, \quad (2.7)$$

with $R = R_{12} + R_{23} + R_{31}$, and conversely,

$$R_{12} = \frac{r}{c}, \quad R_{23} = \frac{r}{a}, \quad R_{31} = \frac{r}{b}, \quad (2.8)$$

where $r = ab + bc + ca$.

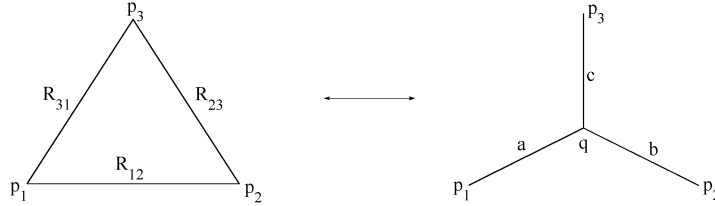


FIGURE 1. Δ – Y transform

In the example in Section 4, we need to use an electrical network with four terminals. We give a version of equivalent electrical networks similar to the Δ -Y transform, and call it the \boxtimes -X transform.

Lemma 2.5. *In the electrical networks as shown in Figure 2, the two networks are equivalent with the following resistances: assuming $yz = x^2$, we have*

$$a = \frac{xy}{2(x+y)}, \quad \text{and} \quad b = \frac{xz}{2(x+z)} \left(= \frac{x^2}{2(x+y)} \right),$$

and equivalently,

$$x = 2(a + b), \quad y = \frac{2a}{b}(a + b), \quad \text{and} \quad z = \frac{2b}{a}(a + b).$$

Proof. We only outline the identity for a . By using the Δ -Y transform on the square together with $\overline{p_2 p_4}$, it is easy to calculate the resulting resistances of p_1 to p_3 is x (this can also be obtained by observing that no current should pass through $\overline{p_2 p_4}$). Then take this in parallel with the resistance y on $\overline{p_1 p_3}$, we have the desired expression. \square

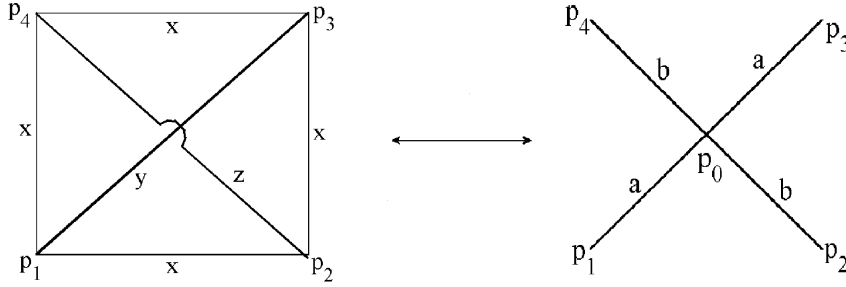


FIGURE 2. equivalent networks with four vertices

3. Besov spaces on p.c.f. sets

In this section, we will provide a discrete expression of (1.2) on a p.c.f. self-similar set K . The main idea of proof is similar to Jonsson [10] on the Sierpinski gasket. We first recall a result from [6].

Proposition 3.1. *For $2\sigma > \alpha$, the identity map $\iota : B_{2,\infty}^\sigma \rightarrow C^{(2\sigma-\alpha)/2}(K)$ is a continuous embedding. (Here $C^\beta(K)$ denotes the class of Hölder continuous functions of order β on K .)*

(Note that in [6], the proposition is stated under the assumption that a heat kernel exists, but it was not used in the proof.) It follows that for $2\sigma > \alpha$, all the functions in $B_{2,\infty}^\sigma$ are continuous.

It is easy to see that the semi-norm $[u]_{B_{2,\infty}^\sigma}$ has an equivalent expression

$$[u]_{B_{2,\infty}^\sigma}^2 \asymp \sup_{j \geq 0} r^{-(2\sigma+\alpha)j} \int_K \int_{B(x, cr^j)} |u(x) - u(y)|^2 d\mu(y) d\mu(x), \quad (3.1)$$

where j is an integer, and $c > 0, 0 < r < 1$ are any fixed constants. Let $V_n = \bigcup \{V_\omega : |\omega| = n\}$ and let $\mu_n = \frac{1}{|V_n|} \sum_{p \in V_n} \delta_p$ where δ_p is the Dirac measure at point p , then $\mu_n \rightarrow \mu$ weakly. Let

$$I_{n,j} = \int_K \int_{B(x, cr^j)} |u(x) - u(y)|^2 d\mu_n(y) d\mu_n(x).$$

Lemma 3.2. *With the above notations, we have*

$$\lim_{n \rightarrow \infty} I_{n,j} = \int_K \int_{B(x, cr^j)} |u(x) - u(y)|^2 d\mu(y) d\mu(x).$$

Proof. It is well-known that if ν_n converges to μ weakly on a compact Hausdorff space E , then $\lim_{n \rightarrow \infty} \int_E f d\nu_n = \int_E f d\mu$ for any Borel measurable f such that the set of discontinuity of f is a μ -zero set. By Proposition 3.1, u is continuous; letting $f(x, y) = \chi_{B(x, cr^j)}(y) \cdot |u(x) - u(y)|^2$, then the set of discontinuity points of f is the set $E = \bigcup_{x \in K} (\{x\} \times \partial B(x, cr^j))$. By Proposition 3.3 in the following, $\mu(\partial B(x, r)) = 0$, so that $(\mu \times \mu)(E) = 0$. Hence the weak convergence applies, and the lemma follows. \square

The following proposition which is used in the above proof is kindly provided by D.J. Feng. It makes use of a result due to Elekes, Keleti and Máthé [4]: *Let μ be a self-similar measure on \mathbb{R}^n . Suppose that μ is not an atom, then for any affine subspace A of \mathbb{R}^d , either $\mu(A \cap K) = 0$ or $A \supseteq K$, where K is the self-similar set that supports μ .*

Proposition 3.3. (Feng [5]) *For any IFS $\{F_i\}_{i=1}^N$ of similitudes on \mathbb{R}^d , let μ be a self-similar measure supported by K , and suppose that K is not contained in a hyperplane, then for any ball $B(x, r) \subset \mathbb{R}^d$, $\mu(\partial B(x, r)) = 0$.*

Proof. Suppose the lemma does not hold. Then there exists a ball $B(x, r) \subset \mathbb{R}^d$ such that $\mu(\partial B(x, r)) > 0$. Let $\delta := \mu(\partial B(x, r)) > 0$. By definition, $\mu = \sum_{i=1}^N p_i \cdot \mu \circ F_i^{-1}$, where $p_i > 0$ and $\sum_{i=1}^N p_i = 1$. Hence for $n > 0$,

$$\mu(\partial B(x, r)) = \sum_{|\omega|=n} p_\omega \cdot \mu(F_\omega^{-1}(\partial B(x, r))).$$

It follows that for any $n > 0$, there exists a word ω_n with $|\omega_n| = n$ such that

$$\mu(F_{\omega_n}^{-1}(\partial B(x, r))) \geq \delta.$$

As $F_{\omega_n}^{-1}(\partial B(x, r)) = \partial B(F_{\omega_n}^{-1}(x), \rho^{-n}r)$, the radii of the balls increases to ∞ as $n \rightarrow \infty$. Let $B(0, R)$ be a ball containing K . Note that for n large, if $F_{\omega_n}^{-1}(\partial B(x, r)) \cap B(0, R) \neq \emptyset$, then by the smoothness of the sphere, there exists a hyperplane E such that the Hausdorff distance of $F_{\omega_n}^{-1}(\partial B(x, r)) \cap$

$B(0, R)$ and $E \cap B(0, R)$ is small. By the compactness of the family of compact sets in the Hausdorff metric, there exists a sequence (n_k) and a hyperplane H ,

$$F_{\omega_{n_k}}^{-1}(\partial B(x, r)) \cap B(0, R) \rightarrow H \cap B(x, R).$$

This implies

$$\mu(H) \geq \liminf_{n \rightarrow \infty} \mu(F_{\omega_{n_k}}^{-1}(\partial B(x, r))) \geq \delta.$$

By the assertion before the proposition, we conclude that $H \supset K$. This is a contradiction. \square

Theorem 3.4. *Suppose the IFS $\{F_i\}_{i=1}^N$ is as in (2.1), and has the p.c.f. property. Then for $2\sigma > \alpha$, we have for $u \in B_{2,\infty}^\sigma$,*

$$[u]_{B_{2,\infty}^\sigma}^2 \asymp \sup_{j \geq 0} \left\{ \rho^{-(2\sigma-\alpha)j} \sum_{x,y \in V_\omega, |\omega|=j} |u(x) - u(y)|^2 \right\}.$$

Proof. First we fix the r in (3.1) to be ρ , the contraction ratio of $\{F_i\}_{i=1}^N$, and let C_0 in Proposition 2.2 to be the c in $I_{n,j}$. We will prove there exists $C > 0$ such that for any $0 < j < n$,

$$I_{n,j} \leq C \rho^{2\alpha j} \sum_{k=j}^n \sum_{x,y \in V_\omega, |\omega|=k} |u(x) - u(y)|^2. \quad (3.2)$$

Then

$$\begin{aligned} \rho^{-(2\sigma+\alpha)j} I_{n,j} &\leq C \rho^{-(2\sigma-\alpha)j} \left(\sum_{k=j}^n \rho^{(2\sigma-\alpha)k} \right) \left(\sup_{k \geq 0} \rho^{-(2\sigma-\alpha)k} \sum_{x,y \in V_\omega, |\omega|=k} |u(x) - u(y)|^2 \right) \\ &\leq C' \sup_{k \geq 0} \rho^{-(2\sigma-\alpha)k} \sum_{x,y \in V_\omega, |\omega|=k} |u(x) - u(y)|^2. \end{aligned}$$

By letting $n \rightarrow \infty$, Lemma 3.2 implies that the ' \lesssim ' side of the theorem holds.

To prove (3.2), we let \mathcal{F}_j be the family of all the cells K_ω with word length $|\omega| = j$. By Proposition 2.2, we have $|x - y| \leq c\rho^j$ implies x, y lie in the same or neighboring j -cells. Therefore for $n > j$,

$$\begin{aligned} I_{n,j} &\leq \sum_{\substack{S, S' \in \mathcal{F}_j \\ S \cap S' \neq \emptyset}} \int_S \int_{S'} |u(x) - u(y)|^2 d\mu_n(y) d\mu_n(x) \\ &= \sum_{\substack{S, S' \in \mathcal{F}_j \\ S \cap S' \neq \emptyset}} \sum_{x \in S \cap V_n} \sum_{y \in S' \cap V_n} \frac{1}{|V_n|^2} |u(x) - u(y)|^2 \end{aligned}$$

$$\leq 2 \sum_{\substack{S, S' \in \mathcal{F}_j \\ S \cap S' \neq \emptyset}} \sum_{x \in S \cap V_n} \sum_{y \in S' \cap V_n} \frac{1}{|V_n|^2} (|u(x) - u(z)|^2 + |u(z) - u(y)|^2),$$

where z is any common vertex of S and S' . As $\#(S \cap V_n) = \#(S' \cap V_n) \asymp N^{n-j}$, and $|V_n| \asymp N^n$, by separating the above sum into two parts on x and y and add separately, we have

$$\begin{aligned} I_{n,j} &\leq C \sum_{S \in \mathcal{F}_j} \sum_{z \in S \cap V_j} \sum_{x \in S \cap V_n} \frac{N^{n-j}}{|V_n|^2} |u(x) - u(z)|^2 \\ &\leq C_1 N^{-(n+j)} \sum_{S \in \mathcal{F}_j} \sum_{z \in S \cap V_j} \sum_{x \in S \cap V_n} |u(x) - u(z)|^2. \end{aligned} \quad (3.3)$$

In order to estimate $|u(x) - u(z)|^2$, we see that there exists a decreasing sequence of cells $\{K_{\omega_k}\}_{k=j}^n$ such that $|\omega_k| = k$ with $z \in K_{\omega_j} \cap V_j$, $x \in K_{\omega_n} \cap V_n$. Choose a sequence of vertices $\{z = x_j, x_{j+1}, \dots, x_n = x\}$ such that $x_k \in K_{\omega_k} \cap V_k$ for $k = j, \dots, n$. By Cauchy-Schwarz inequality, we have

$$|u(z) - u(x)|^2 \leq \left(\sum_{k=j}^{n-1} N^{-(k-j)} \right) \left(\sum_{k=j}^{n-1} N^{k-j} |u(x_k) - u(x_{k+1})|^2 \right) \leq C_2 \sum_{k=j}^{n-1} N^{k-j} |u(x_k) - u(x_{k+1})|^2.$$

Using this, and observing that for each $S \in \mathcal{F}_j$, the cardinality of a fixed pair $(p, q) := (x_k, x_{k+1})$ appeared in the summation over $z \in S \cap V_j$, $x \in S \cap V_n$ is $\lesssim N^{n-k}$, we can continue the estimation in (3.3), for $S \in \mathcal{F}_j$,

$$\sum_{z \in S \cap V_j} \sum_{x \in S \cap V_n} |u(x) - u(z)|^2 \leq C_3 \sum_{k=j}^{n-1} \sum_{\substack{|\omega|=k \\ K_\omega \subset S}} \sum_{p, q \in K_\omega \cap V_{k+1}} N^{k-j} \cdot N^{n-k} |u(p) - u(q)|^2$$

As $\rho^\alpha = N^{-1}$, the above inequality gives

$$I_{n,j} \leq C_3 \rho^{2\alpha j} \sum_{k=j}^{n-1} \sum_{S \in \mathcal{F}_j} \sum_{\substack{|\omega|=k \\ K_\omega \subseteq S}} \sum_{p, q \in K_\omega \cap V_{k+1}} |u(p) - u(q)|^2,$$

then for each pair $p, q \in K_\omega \cap V_{k+1}$ with $|\omega| = k$, using a chain of finite points connecting p, q such that every neighboring two points belong to the same $K_\tau \cap V_{k+1}$ with some $|\tau| = k+1$, thus we have

$$\begin{aligned} I_{n,j} &\leq C_4 \rho^{2\alpha j} \sum_{k=j}^{n-1} \sum_{x, y \in V_\tau; |\tau|=k+1} (u(x) - u(y))^2 \\ &\leq C_4 \rho^{2\alpha j} \sum_{k=j}^n \sum_{x, y \in V_\omega; |\omega|=k} (u(x) - u(y))^2. \end{aligned}$$

This completes the proof of (3.2).

Next we prove the reverse inequality of the theorem. We use the notation $\int_E u(x) d\mu(x)$ to denote $\frac{1}{\mu(E)} \int_E u(x) d\mu(x)$, and observe that for any ω , we have

$$\begin{aligned} \sum_{p, q \in V_\omega, |\omega|=j} |u(p) - u(q)|^2 &\leq 2 \sum_{p, q \in V_\omega, |\omega|=j} \int_{K_\omega} (|u(p) - u(x)|^2 + |u(x) - u(q)|^2) d\mu(x) \\ &\leq 4|V_\omega| \sum_{p \in V_\omega, |\omega|=j} \int_{K_\omega} |u(p) - u(x)|^2 d\mu(x). \end{aligned} \quad (3.4)$$

For each $p \in V_\omega$ with $|\omega| = j$, there is a uniquely determined decreasing sequence of cells $\{K_{\omega_k}\}_{k=j}^n$ such that $p \in \bigcap_{k=j}^n K_{\omega_k}$ with $\omega_j = \omega$, $|\omega_{k+1}| = |\omega_k| + \ell$ for $k = j, \dots, n-1$, where n and ℓ are two positive integers sufficiently large to be determined later. By using this and the Cauchy-Schwarz inequality, we have

$$|u(p) - u(x_j)|^2 \leq 2|u(p) - u(x_n)|^2 + 2\left(\sum_{k=j}^{n-1} \rho^{k-j}\right)\left(\sum_{k=j}^{n-1} \rho^{-(k-j)} |u(x_k) - u(x_{k+1})|^2\right)$$

where $x_k \in K_{\omega_k}$. Continue the estimate in (3.4), we have

$$\begin{aligned} \int_{K_\omega} |u(p) - u(x_j)|^2 d\mu(x_j) &\leq C \int_{K_{\omega_n}} |u(p) - u(x_n)|^2 d\mu(x_n) \\ &\quad + C \sum_{k=j}^{n-1} \rho^{-(k-j)} \cdot \int_{K_{\omega_k}} \int_{K_{\omega_{k+1}}} |u(x_k) - u(x_{k+1})|^2 d\mu(x_{k+1}) d\mu(x_k), \end{aligned} \quad (3.5)$$

Combining (3.4) and the second term on the right side of (3.5), and observe that $|x_k - x_{k+1}| \leq \rho^{j+(k-j)\ell}$ and $\mu(K_{\omega_k}) \asymp \mu(K_{\omega_{k+1}}) \asymp \rho^{(j+(k-j)\ell)\alpha}$. We have

$$\begin{aligned} &\rho^{-(2\sigma-\alpha)j} \sum_{p \in V_\omega, |\omega|=j} \left(\sum_{k=j}^{n-1} \rho^{-(k-j)} \int_{K_{\omega_k}} \int_{K_{\omega_{k+1}}} |u(x_k) - u(x_{k+1})|^2 d\mu(x_{k+1}) d\mu(x_k) \right) \\ &\leq C_1 \rho^{-(2\sigma-\alpha)j} \sum_{k=j}^{n-1} \rho^{-(k-j)} \rho^{-2(j+(k-j)\ell)\alpha} \int_K \int_{|x_k - x_{k+1}| \leq \rho^{j+(k-j)\ell}} |u(x_k) - u(x_{k+1})|^2 d\mu(x_{k+1}) d\mu(x_k) \\ &\leq C_2 \rho^{-(2\sigma-\alpha)j} \sum_{k=j}^{n-1} \rho^{-(k-j)} \rho^{-2(j+(k-j)\ell)\alpha} \rho^{(j+(k-j)\ell)(2\sigma+\alpha)} \cdot [u]_{B_{2,\infty}^\sigma}^2 \\ &\leq C_2 \sum_{k'=0}^{n-j} \rho^{-k'(\ell(\alpha-2\sigma)+1)} \cdot [u]_{B_{2,\infty}^\sigma}^2 \leq C_3 [u]_{B_{2,\infty}^\sigma}^2, \end{aligned}$$

where ℓ is sufficiently large such that $\ell(\alpha - 2\sigma) + 1 < 0$. For the first term on the right side of (3.5), we have, by Proposition 3.1, for all $x_n \in K_{\omega_n}$,

$$|u(p) - u(x_n)|^2 \leq C|p - x_n|^{2\sigma-\alpha} [u]_{B_{2,\infty}^\sigma}^2 \leq C\rho^{(j+(n-j)\ell)(2\sigma-\alpha)} [u]_{B_{2,\infty}^\sigma}^2.$$

Combining with (3.4), and using a similar argument as the above, choosing n sufficiently large, we have

$$\rho^{-(2\sigma-\alpha)j} \sum_{p \in V_\omega, |\omega|=j} \int_{K_{\omega_n}} |u(p) - u(x_n)|^2 d\mu(x_n) \leq C[u]_{B_{2,\infty}^\sigma}^2,$$

The ' \gtrsim ' part follows from the above two estimations. This completes our proof of the theorem. \square

It follows easily from the above theorem that

Corollary 3.5. *For $2\sigma > \alpha$, and fixed integer $\ell \geq 1$, we have for any $u \in B_{2,\infty}^\sigma$,*

$$\|u\|_{B_{2,\infty}^\sigma}^2 \asymp \sup_{n \geq 0} \left(\rho^{-(2\sigma-\alpha)n\ell} \sum_{x,y \in V_\omega; |\omega|=n\ell} |u(x) - u(y)|^2 \right) + \|u\|_2^2.$$

The following property will be used in the construction of functions in $B_{2,\infty}^\sigma$.

Proposition 3.6. *Assume $2\sigma > \alpha$, then for any function u on V_* , if*

$$[u]_{B_{2,\infty}^\sigma(V_*)}^2 := \sup_{j \geq 0} \left\{ \rho^{-(2\sigma-\alpha)j} \sum_{x,y \in V_\omega, |\omega|=j} |u(x) - u(y)|^2 \right\} < \infty,$$

then u can be extended continuously to \tilde{u} on K , and $\tilde{u} \in B_{2,\infty}^\sigma$.

Proof. The proof is based on the following inequality: there exists $C > 0$ such that for any function u on V_* , and any $x, y \in V_*$,

$$|u(x) - u(y)|^2 \leq C|x - y|^{2\sigma-\alpha} [u]_{B_{2,\infty}^\sigma(V_*)}^2. \quad (3.6)$$

Then u is uniformly continuous on V_* which is dense in K . Hence it has a continuous extension to K .

We turn to prove (3.6). Given $x, y \in V_*$, let k be the integer such that

$$C_0\rho^{k+1} \leq |x - y| < C_0\rho^k,$$

where C_0 is the constant in condition (H). Let ω_1 and ω_2 be two finite words such that $|\omega_1| = |\omega_2| = k$ and $x \in K_{\omega_1}$, $y \in K_{\omega_2}$. Observing that

$$\text{dist}(K_{\omega_1}, K_{\omega_2}) \leq |x - y| < C_0\rho^k,$$

and using condition (H), we see that

$$K_{\omega_1} \cap K_{\omega_2} \neq \emptyset.$$

Let $z \in K_{\omega_1} \cap K_{\omega_2}$, we can find a decreasing sequence of cells starting with ω_1 and a chain of points $x = x_0, x_1, \dots, x_n = z$ such that $x_i \in V_{\omega_i}$, and $\{V_{\omega_i}\}$ is a monotonic sequence as in the proof of Theorem 3.4. Then

$$\begin{aligned} |u(x) - u(z)| &\leq \sum_{i=1}^n |u(x_{i-1}) - u(x_i)| \leq C \sum_{i=1}^n \left(\sum_{p, q \in V_{\omega_i}, |\omega_i|=k+i} |u(p) - u(q)|^2 \right)^{1/2} \\ &\leq C \sum_{i=1}^n \rho^{\frac{(2\sigma-\alpha)(k+i)}{2}} [u]_{B_{2,\infty}^\sigma(V_*)} \leq C \rho^{\frac{k(2\sigma-\alpha)}{2}} [u]_{B_{2,\infty}^\sigma(V_*)}. \end{aligned}$$

We get the same bound for $|u(y) - u(z)|$. Hence

$$|u(x) - u(y)| \leq |u(x) - u(z)| + |u(y) - u(z)| \leq 2C \rho^{\frac{k(2\sigma-\alpha)}{2}} [u]_{B_{2,\infty}^\sigma(V_*)} \leq 2C |x - y|^{\frac{2\sigma-\alpha}{2}} [u]_{B_{2,\infty}^\sigma(V_*)},$$

and proves (3.6). \square

4. Two constructions of Dirichlet forms

In this section, we construct a p.c.f. similar set K such that $B_{2,\infty}^{\sigma*}(\subset C(K))$ is not dense in $C(K)$, but dense in $L^2(K, \mu)$. Hence this Besov space does not have the regularity property as the domain of a Dirichlet form. On the other hand, we can construct two different types of local regular Dirichlet forms on K (with different domains), one satisfies the energy self-similar identity [12], the other one does not.

In \mathbb{R}^2 , let $\{p_1, p_2, p_3, p_4\}$ be the four vertices of the unit square S , and let p_0 be the center of S , that is, $p_0 = (0, 0)$ and $p_1 = (-1/2, -1/2)$, $p_2 = (1/2, -1/2)$, $p_3 = (1/2, 1/2)$, $p_4 = (-1/2, 1/2)$. Divide S into a mesh of sub-squares of size $1/9$, and pick 21 sub-squares as shown in Figure 3.

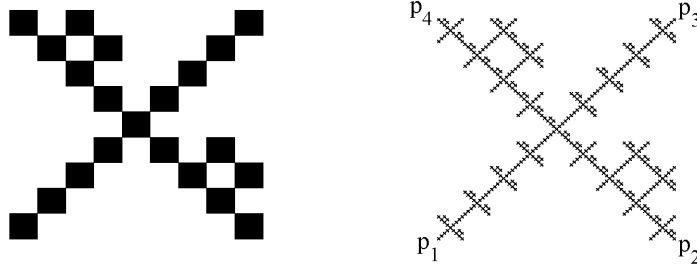


FIGURE 3. The Vicsek eyebolted cross K

Let $\{a_i\}_{i=1}^{21}$ be the center of these sub-squares. Let $\{F_i\}_{i=1}^{21}$ be the IFS on \mathbb{R}^2 with

$$F_i(x) = \frac{1}{9}(x - a_i) + a_i, \quad 1 \leq i \leq 21,$$

where $F_i, 1 \leq i \leq 9$ corresponds to the 9 sub-squares along $\overline{p_1 p_3}$, and the other F_i corresponds to the other 12 sub-squares; let K be the unique nonempty compact set such that $K = \bigcup_{i=1}^{21} F_i(K)$. Then $(K, \{F_i\}_{i=1}^{21})$ is a p.c.f. self-similar set with boundary $V_0 = \{p_1, p_2, p_3, p_4\}$. We call this modified Vicsek cross a *Vicsek eyebolted cross*. The Hausdorff dimension of K is $\alpha = \log 21 / \log 9$, and the self-similar measure with the natural weight is the normalized α -dimensional Hausdorff measure μ on K .

4.1. Critical exponents. We will make use of the \boxtimes -X transform in Lemma 2.5. For $n \geq 0$, let G_n denote the graph on V_n with edges connecting any two vertices on each subcell $F_\omega(V_0)$ (i.e., there are 6 edges on $F_\omega(V_0)$ as in the left picture in Figure 2). Let $V'_0 = V_0 \cup \{p_0\}$, and let G'_0 be the graph of G_0 by adding one more vertex p_0 as in the right picture of Figure 2. We define the graph G'_n such that each subcell has the graph $F_\omega(G'_0)$, $|\omega| = n$.

Lemma 4.1. *Suppose G'_1 has resistance $\xi = (a, b, c, d)$ on the four edges of each subcell. Let $\Phi(\xi) = (a', b', c', d')$ be the resulting resistance of G'_1 on V_0 , then*

$$\Phi(\xi) = (5a + 4c, 4b + 3d + \varphi(\xi), 4a + 5c, 3b + 4d + \varphi(\xi)). \quad (4.1)$$

where $\varphi(\xi) = \frac{(b+d)(2a+2c+b+d)}{2(a+b+c+d)}$.

Proof. The expression of $\varphi(\xi)$ is obtained by applying the parallel law of the resistance to the two squares on the graph G'_1 (see Figure 4). That $a' = 5a + 4c$ is by applying the series law to the branch of p_1 to the center, and the same for b', c', d' on the other three branches. \square

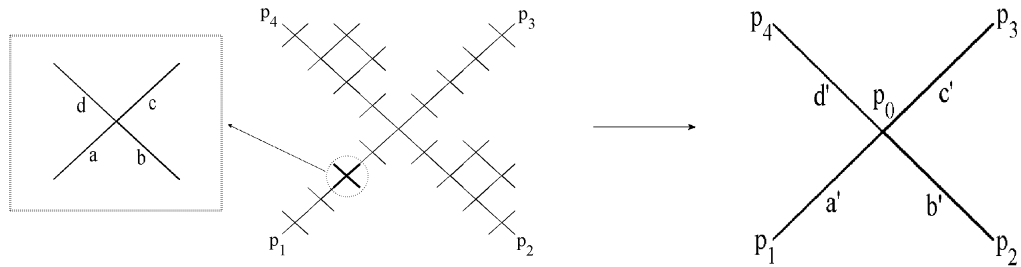


FIGURE 4. Network on G'_1 and its resulting resistances on V'_0

Proposition 4.2. *Suppose each edge of G_n has resistance 1. Let $R_n(p, q)$, $p, q \in V_0$ be the resulting resistance of G_n on V_0 , then we have*

$$\begin{aligned} R_n(p_1, p_2) &= R_n(p_2, p_3) = R_n(p_3, p_4) = R_n(p_4, p_1) = \frac{a_n + b_n}{2} \asymp a_n; \\ R_n(p_1, p_3) &= \frac{1}{2}\left(a_n + \frac{a_n^2}{b_n}\right) \asymp \frac{a_n^2}{b_n}; \\ R_n(p_2, p_4) &= \frac{1}{2}\left(b_n + \frac{b_n^2}{a_n}\right) \asymp b_n, \end{aligned}$$

where $a_n = 9^n$ and $b_n = 9b_{n-1} - \frac{b_{n-1}^2}{9^{n-1} + b_{n-1}}$ with $b_0 = 1$ for $n \geq 1$.

Remark. Note that $\lim_{n \rightarrow \infty} b_n/9^n = 0$, and for any $0 < \varepsilon < 9$, $\lim_{n \rightarrow \infty} b_n/(9 - \varepsilon)^n = \infty$.

Proof. First we use the \boxtimes -X transform to convert the resistances from G_n to G'_n , it follows that each cell in G'_n has resistance $\xi = \frac{1}{4}(1, 1, 1, 1)$ (Lemma 2.5). By applying Lemma 4.1, we obtain the resulting resistance $\Phi(\xi)$ on each cell of G'_{n-1} . By induction and some simple calculation, we see that the resulting resistance of G'_n on G'_0 is given by

$$\Phi^n(\xi) = \frac{1}{4}(a_n, b_n, a_n, b_n),$$

where $a_n = 9^n$ and $b_n = 9b_{n-1} - \frac{b_{n-1}^2}{9^{n-1} + b_{n-1}}$, ($b_0 = 1$). Then applying the inverse \boxtimes -X transform (Lemma 2.5), we obtain the expressions of $R_n(p, q)$ for $p, q \in V_0$ as stated; the asymptotic values follows from the remark. \square

Theorem 4.3. *For the Besov spaces $B_{2,\infty}^\sigma$ defined on the Vicsek eyebolted cross K , the critical exponents are*

$$\sigma^* = \sigma^\# = \frac{1}{2}\left(1 + \frac{\log 21}{\log 9}\right).$$

Moreover, $B_{2,\infty}^{\sigma^*}$ is dense in $L^2(K, \mu)$, but not dense in $C(K)$.

Proof. For $u \in V_n$ with fixed values $u(p_i)$, $i = 1, 2, 3, 4$, by the resulting resistance of G_n on V_0 , we have

$$\begin{aligned} \min_u \left\{ \sum_{x,y \in V_\omega; |\omega|=n} |u(x) - u(y)|^2 \right\} &= \sum_{i \neq j} \frac{1}{R_n(p_i, p_j)} |u(p_i) - u(p_j)|^2 \\ &\asymp \frac{1}{a_n} \sum_{i=1}^4 |u(p_i) - u(p_{i+1})|^2 + \frac{b_n}{a_n^2} |u(p_1) - u(p_3)|^2 + \frac{1}{b_n} |u(p_2) - u(p_4)|^2 \end{aligned} \quad (4.2)$$

(here $p_5 = p_1$ as convention). Multiply $a_n = 9^n$ to both sides of (4.2), we obtain,

$$\min_u \left\{ 9^n \sum_{x,y \in V_\omega; |\omega|=n} |u(x) - u(y)|^2 \right\} \asymp \sum_{i=1}^4 |u(p_i) - u(p_{i+1})|^2 + \frac{b_n}{a_n} |u(p_1) - u(p_3)|^2 + \frac{a_n}{b_n} |u(p_2) - u(p_4)|^2. \quad (4.3)$$

It follows that for any $\varepsilon > 0$, $\min_u \left\{ 9^{(1+\varepsilon)n} \sum_{x,y \in V_\omega; |\omega|=n} |u(x) - u(y)|^2 \right\}$ is unbounded on n for any non-identical $u(p_i), i = 1, 2, 3, 4$. Note that the same argument apply to any sub-cell K_ω in K .

As the contraction ratio of the IFS is $\rho = 9^{-1}$, the Hausdorff dimension of K is $\alpha = \frac{\log 21}{\log 9}$. For $2\sigma > \alpha + 1$, by Theorem 3.4 and regarding $\rho^{-(2\sigma-\alpha)}$ as $9^{(1+\varepsilon)n}$ as the above, we conclude that $B_{2,\infty}^\sigma$ can only contain constant functions. Hence $\sigma^\# \leq (\alpha + 1)/2$.

Next we consider $2\sigma < \alpha + 1$. For any $u(p_i), i = 1, 2, 3, 4$, by (4.3) and the estimation of a_n/b_n in the remark of Proposition 4.2, we can find an $N = N(\sigma)$ such that

$$(9^{2\sigma-\alpha})^N \sum_{x,y \in V_\omega, |\omega|=N} |u(x) - u(y)|^2 \leq \sum_{x,y \in V_0} |u(x) - u(y)|^2.$$

Set u to be the minimal energy function on V_N , and continuing this procedure on V_{2N}, V_{3N}, \dots , there is u on $V_* = \bigcup_{n \geq 0} V_n$ such that all $k \geq 1$,

$$9^{(2\sigma-\alpha)kN} \sum_{x,y \in V_\omega, |\omega|=kN} |u(x) - u(y)|^2 \leq \sum_{x,y \in V_0} |u(x) - u(y)|^2.$$

By Corollary 3.5 and Proposition 3.6, u can be extended continuously to K and $u \in B_{2,\infty}^\sigma$. It follows that for any $v \in C(K)$, if we let v_n to be the restriction of v on V_n , we can extend v_n on each cell $K_\omega, |\omega| = n$. Then $v_n \in B_{2,\infty}^\sigma$, and $\{v_n\}_{n=1}^\infty$ converges to v uniformly. This shows that $B_{2,\infty}^\sigma$ is dense in $C(K)$. Therefore $(\alpha+1)/2 \leq \sigma^*$. In conclusion, we have $\frac{1}{2}(\alpha+1) \leq \sigma^* \leq \sigma^\# \leq \frac{1}{2}(\alpha+1)$, and the first part follows.

For the second part, we consider $B_{2,\infty}^\sigma$ at σ^* . We see from (4.3) and $\lim_{n \rightarrow \infty} a_n/b_n = \infty$ that any $u \in B_{2,\infty}^{\sigma^*}$ must satisfy $u(p_2) = u(p_4)$, this implies that $B_{2,\infty}^{\sigma^*}$ is not dense in $C(K)$.

To show that $B_{2,\infty}^{\sigma^*}$ is dense in $L^2(K, \mu)$, for any $\varepsilon > 0$, and any $f \in L^2(K, \mu)$, let $g \in C(K)$ such that $\|g - f\|_2 \leq \frac{\varepsilon}{2}$. Let m be the integer such that $|g(x) - g(y)| \leq \frac{\varepsilon}{2\sqrt{8\mu(K)}}$ for any $x, y \in K$ with $|x - y| \leq 2\sqrt{2} \cdot 9^{-m}$. Let L denote the line segment $\overline{p_2 p_4} \subset K$. Let $P_m = \bigcup \{F_\omega(L) : |\omega| = m\}$. For $n > m$, let $E_n = \bigcup_{|\omega|=n, K_\omega \cap P_m \neq \emptyset} K_\omega$, and choose an n large so that $\mu(E_n) \leq \frac{\varepsilon^2}{32\|g\|_\infty^2}$.

We can write $P_n = \bigcup_k L_n^k$ where L_n^k are maximal line segments in P_n parallel to L . Define g_n on V_n such that

$$g_n(x) = \begin{cases} \max_{y \in V_n \cap L_n^k} g(y), & x \in L_n^k \cap V_n; \\ g(x), & x \in V_n \setminus P_n. \end{cases}$$

We see that on each cell V_ω with $|\omega| = n$, $g_n(F_\omega(p_2)) = g_n(F_\omega(p_4))$. We can extend g_n to V_* as follows: First we extend g_n to V_{n+1} . Note that for each $|\omega| = n$, $F_\omega(V_0)$ has four branches in V_{n+1} (see Figure 3 as $F_\omega(V_0)$, and $\bigcup_i F_{\omega i}(V_0)$). Assign the constant value $g_n(F_\omega(p_2))$ on the $F_\omega(p_2)$ and $F_\omega(p_4)$ branches (include the center point $F_\omega(p_0)$), and set g_n to be linear on the other two branches $F_\omega(p_1)$ and $F_\omega(p_3)$. Continuing this process inductively, we obtain a continuous extension of g_n on V_* , it is constant in the $\overline{p_2 p_4}$ direction, and piecewise linear in the $\overline{p_1 p_3}$ direction. Also from the construction, g_n can be extended continuously to K , still denote by g_n , and is in $B_{2,\infty}^{\sigma^*}$. Now consider the m, n chosen previously, we have

$$\begin{aligned} \|g_n - g\|_2^2 &= \int_{E_n} |g_n - g|^2 d\mu + \int_{K \setminus E_n} |g_n - g|^2 d\mu \\ &\leq (2\|g\|_\infty)^2 \cdot \mu(E_n) + \frac{\varepsilon^2}{8\mu(K)} \cdot \mu(K) \leq \frac{\varepsilon^2}{4}. \end{aligned}$$

This implies $\|g_n - f\|_2 \leq \|g_n - g\|_2 + \|g - f\|_2 \leq \varepsilon$. The denseness of $B_{2,\infty}^{\sigma^*}$ in $L^2(K, \mu)$ follows. \square

4.2. Dirichlet form. Since $B_{2,\infty}^{\sigma^*}$ is not dense in $C(K)$, $B_{2,\infty}^{\sigma^*}$ cannot be the domain of a local regular Dirichlet form. Nevertheless we will give two constructions of such Dirichlet forms on K such that the domains are different from $B_{2,\infty}^{\sigma^*}$.

Theorem 4.4. *On the Vicsek eyebolted cross, there are two kinds of local regular Dirichlet forms that can be constructed on K , one satisfies the energy self-similar identity (2.6), and the other one does not.*

Proof. First construction: We will assign two different renormalizing factors τ', τ'' (to be determined) on the cells of K as follows: let $\tau_1 = \tau_2 = \dots = \tau_9 = \tau'$ on the 9 sub-cells $F_1(K), \dots, F_9(K)$ along the line $\overline{p_1 p_3}$, and let $\tau_{10} = \tau_{11} = \dots = \tau_{21} = \tau''$ on the remaining 12 sub-cells $F_{10}(K), \dots, F_{21}(K)$; then similar to Lemma 4.1, we obtain a new resulting resistance map Φ_{τ_U, τ_D} for $\xi = (a, b, c, d)$:

$$\Phi_{\tau', \tau''}(\xi) = (\tau'(5a + 4c), \tau''(3b + 3d + \varphi(\xi)) + \tau'b, \tau'(4a + 5c), \tau''(3b + 3d + \varphi(\xi)) + \tau'd).$$

where $\varphi(\xi) = \frac{(b+d)(2a+2c+b+d)}{2(a+b+c+d)}$. Consider the equation

$$\Phi_{\tau', \tau''}(a, b, c, d) = (a, b, c, d), \quad (4.4)$$

i.e., the resulting resistance of G'_1 coincides with the resistances on G'_0 . If we apply this to G'_n inductively, then we obtain a sequence of networks $\{G'_k\}_{k=0}^n$ that is compatible in the sense of Definition 2.4 (see also [12, Definition 2.1.10]), and given the energy self-similar identity. Specifically, let us take $a = b = c = d$ in (4.4), then it reduces to be two simple linear equations, and the solution is

$$\tau' = \frac{1}{9}, \quad \tau'' = \frac{16}{135}.$$

Let $E_0(u) = \sum_{p,q \in V_0} (u(p) - u(q))^2$, define

$$\mathcal{E}[u] = \lim_{n \rightarrow \infty} \sum_{|\omega|=n} \tau_\omega^{-1} E_0(u \circ F_\omega),$$

and $\mathcal{E}[u] < \infty$ implies that $u \in C(K)$ (e.g. [12, Theorem 2.2.6(1)]), thus we can let $\mathcal{F} = \{u \in C(K) : \mathcal{E}[u] < \infty\}$. Then $(\mathcal{E}, \mathcal{F})$ satisfies the self-similar identity

$$\mathcal{E}[u] = \sum_{i=1}^{17} \tau_i^{-1} \mathcal{E}[u \circ F_i], \quad u \in \mathcal{F}.$$

It is known that this defines a regular local Dirichlet form on the metric measure space (K, d_r, ν) , where d_r is the resistance metric on K , and ν is the self-similar measure with weights $\{\tau_i^s\}_{i=1}^N$ where $\sum_{i=1}^N \tau_i^s = 1$ ([7], [9]).

Second construction: The main idea is to use Φ^{-n} (where Φ is defined in (4.1)) to construct a sequence of conductances $\{c_n(x, y)\}_n$ in (2.5) such that $\mathcal{E}_n[u]$ converges for $u \in C(K)$.

Consider the network on G'_n , and let \mathbf{y}_n be the resistance on each cell of G'_n . We are looking for $\mathbf{y}_n = (s_n, t_n, s_n, t_n)$ such that the resulting resistance is $\mathbf{y}_0 = (1, 1, 1, 1)$, i.e., $\Phi^n(\mathbf{y}_n) = \mathbf{y}_0$. As $\Phi(s, t, s, t) = (9s, 9t - \frac{t^2}{s+t}, 9s, 9t - \frac{t^2}{s+t})$, it follows that

$$\mathbf{y}_n = \Phi^{-n}(\mathbf{y}_0) = (s_n, t_n, s_n, t_n)$$

where $s_n = 9^{-n}$ and $t_{n-1} = 9t_n - \frac{t_n^2}{9^{-n} + t_n} (> 0)$ for $n \geq 1$. Hence by the compatibility of G'_n and G'_0 with resistance \mathbf{y}_n and \mathbf{y}_0 respectively, we have

$$\begin{aligned} & \min_v \left\{ \sum_{|\omega|=n} \left(s_n^{-1} \sum_{i=1,3} |v \circ F_\omega(p_i) - v \circ F_\omega(p_0)|^2 + t_n^{-1} \sum_{j=2,4} |v \circ F_\omega(p_j) - v \circ F_\omega(p_0)|^2 \right) \right\} \\ &= \sum_{p \in V_0} |u(p) - u(p_0)|^2. \end{aligned} \quad (4.5)$$

where the minimum is taken over all $v \in \ell(V'_n)$ such that $v|_{V_0} = u$. Then by applying the inverse of \boxtimes -X transform (Lemma 2.5) to each cell in G'_n , we obtain an equivalent network on G_n .

$$\min_v \left\{ \sum_{x,y \in V_\omega, |\omega|=n} c_n(x,y) |v(x) - v(y)|^2 \right\} = \sum_{p,q \in V_0} \frac{1}{4} |u(p) - u(q)|^2, \quad (4.6)$$

where the resistances $c_n(x,y)^{-1}$ on V_n are given by

$$\begin{aligned} c_n(F_\omega(p_i), F_\omega(p_{i+1}))^{-1} &= 2(s_n + t_n), \quad i = 1, 2, 3, 4, \quad (p_5 = p_1) \\ c_n(F_\omega(p_1), F_\omega(p_3))^{-1} &= 2(s_n + \frac{s_n^2}{t_n}), \\ c_n(F_\omega(p_2), F_\omega(p_4))^{-1} &= 2(t_n + \frac{t_n^2}{s_n}). \end{aligned}$$

For $u \in C(K)$ and $n \geq 0$, let

$$\mathcal{E}_n[u] = \sum_{x,y \in V_\omega, |\omega|=n} c_n(x,y) |u(x) - u(y)|^2.$$

By the compatibility of G_n and G_{n-1} through the \mathbf{y}_n and \mathbf{y}_{n-1} , we see that $\mathcal{E}_n[u]$ is an increasing sequence on n , define

$$\mathcal{E}[u] = \lim_{n \rightarrow \infty} \mathcal{E}_n[u], \quad \mathcal{F} = \{u \in C(K) : \mathcal{E}(u) < \infty\}.$$

Note that \mathcal{F} is dense in $C(K)$ by approximating $u \in C(K)$ through piecewise harmonic functions constructed from (4.6) applied to the subcells. Hence it is not hard to see that $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on the metric measure space $(K, |\cdot|, \mu)$.

Finally we show by contradiction that the above Dirichlet form does not satisfy the self-similar identity. Assume that there exist positive numbers $\tau_1, \tau_2, \dots, \tau_{21}$ such that for any $u \in \mathcal{F}$,

$$\mathcal{E}[u] = \sum_{i=1}^{21} \tau_i^{-1} \mathcal{E}[u \circ F_i]. \quad (4.7)$$

Recall that in our construction, the weight we put on each cell is the same, then we have $\tau_1 = \tau_2 = \dots = \tau_{21} = \tau$.

Let u_1 be the function that is linear on the line segment $\overline{p_1 p_3}$ with boundary values $u_1(p_1) = 1$, $u_1(p_3) = 0$, and u_1 is constant on all the eyebolted branches issued at some point on $\overline{p_1 p_3}$. Then the energy of u_1 is supported on $\overline{p_1 p_3}$. We can easily show that $\mathcal{E}_n[u_1] = \frac{1}{2}$ for all $n \geq 0$, and thus $\mathcal{E}[u_1] = \frac{1}{2}$. Similarly we have $\mathcal{E}[u_1 \circ F_i] = \frac{1}{2} \cdot \frac{1}{9^i}$ for each $i = 1, 2, \dots, 9$ along the line $\overline{p_1 p_3}$, and $\mathcal{E}[u \circ F_i] = 0$ for the rest twelve maps. By using (4.7), we obtain $\tau = \frac{1}{9}$.

Let u_2 be the harmonic function with boundary values $(u_2(p_1), u_2(p_2), u_2(p_3), u_2(p_4)) = (0, 1, 0, 0)$. By using (4.7) n times with $\tau_1 = \tau_2 = \cdots = \tau_{21} = 1/9$, and $u = u_2$, we obtain

$$\mathcal{E}[u_2] = \sum_{|\omega|=n} \tau_\omega^{-1} \mathcal{E}[u_2 \circ F_\omega] = 9^n \sum_{|\omega|=n} \mathcal{E}[u_2 \circ F_\omega], \quad n > 0. \quad (4.8)$$

Since for any $u \in \mathcal{F}$, $\mathcal{E}[u]$ is the limit of the increasing sequence $\mathcal{E}_n[u]$, we have

$$9^n \sum_{|\omega|=n} \mathcal{E}[u_2 \circ F_\omega] \geq 9^n \sum_{|\omega|=n} \mathcal{E}_0[u_2 \circ F_\omega] = 9^n \sum_{p, q \in V_\omega, |\omega|=n} \frac{1}{4} |u_2(p) - u_2(q)|^2. \quad (4.9)$$

On the other hand, by (4.3), we have

$$\begin{aligned} 9^n \sum_{p, q \in V_\omega, |\omega|=n} \frac{1}{4} |u_2(p) - u_2(q)|^2 &\geq \frac{1}{4} \min_u \left\{ 9^n \sum_{x, y \in V_\omega, |\omega|=n} |u(x) - u(y)|^2 \right\} \\ &\geq C^{-1} \frac{a_n}{b_n} |u_2(p_2) - u_2(p_4)|^2 \\ &= C^{-1} \frac{a_n}{b_n} \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned} \quad (4.10)$$

Hence we see from (4.8), (4.9) and (4.10) that $\mathcal{E}(u_2) = \infty$, contradicting $u_2 \in \mathcal{F}$. Therefore the Dirichlet form does not satisfy the energy self-similar identity. \square

Remark. In the example, if we lift the lower right eyebolt to the upper right position, then the abnormality of the density in Theorem 4.3 will not appear. We can show that for this new K , $\sigma^* = \sigma^\# = \frac{\log 21 + \log(35/4)}{\log 9}$, the renormalizing factor is $r = \frac{4}{35}$, and the associate Besov space $B_{2,\infty}^{\sigma^*}$ does support a local regular Dirichlet form.

5. An example with $\sigma^* \neq \sigma^\#$

Let $V_0 = \{p_1, p_2, p_3\}$ with $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $\{F_i\}_{i=1}^{17}$ be the IFS of contractive similitudes on \mathbb{R}^2 such that

$$F_i(x) = \frac{1}{7}x + a_i, \quad 1 \leq i \leq 17,$$

where the a_i 's are the 17 points lie on the triangle determined by V_0 as indicated in Figure 5. Let K be the unique nonempty compact set such that $K = \bigcup_{i=1}^{17} K_i$, and call it a *Sierpinski sickle*. Then $K_i \cap K_j$ contains at most one point with the expression $i\dot{k}$, and thus satisfies the p.c.f condition. The Hausdorff dimension of K is $\alpha = \log 17 / \log 7$, and the self-similar measure

with the nature weight is the normalized α -dimensional Hausdorff measure μ on K .

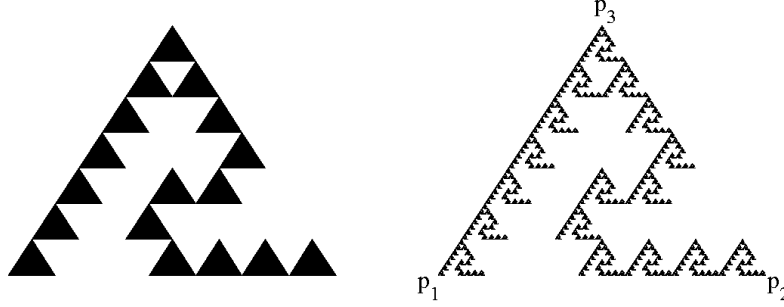


FIGURE 5. The Sierpinski sickle K

5.1. Critical exponents. For $n \geq 0$, let G_n denote the graphs on V_n with edges connecting the vertices of the intersecting cells. On G_0 , we arrange the three edges clockwise in the order of $\overline{p_1 p_2}$, $\overline{p_2 p_3}$, $\overline{p_3 p_1}$, and the same way for the sub-triangles in G_n . Let G'_n be the corresponding graph in the Y -form of the Δ - Y transform.

Lemma 5.1. *Suppose G'_1 has resistance $\xi = (a, b, c)$ on the three edges of the subcells, counting from counterclockwise direction (as indicated in Figure 6). Then the resulting resistance $\Phi(\xi)$ of G'_1 on G'_0 is:*

$$\Phi(\xi) = (a', b', c') = (6a + 5c + \varphi_a, 6a + 8b + 5c + \varphi_b, c + \varphi_c). \quad (5.1)$$

where $\varphi_a = \frac{(a+b)(a+c)}{2(a+b+c)}$, and φ_b, φ_c are defined symmetrically.

Proof. We apply the $\Delta - Y$ transform and obtain (5.1) through a direct calculation (see Figure 6). \square

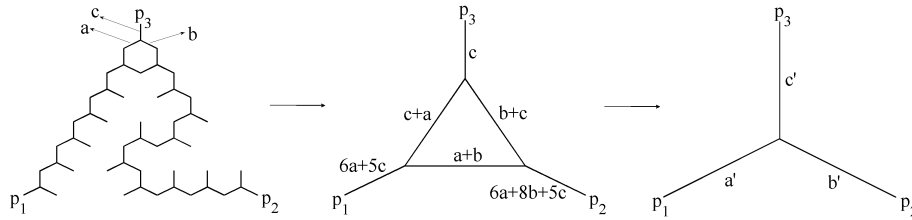


FIGURE 6. The resulting resistance of G'_1 on G'_0

We will apply the above lemma on the graphs G_n inductively. For this we need some simple estimations.

Lemma 5.2. *Let $\Phi(\xi) = (a', b', c')$ be as in (5.1). Then*

- (i) *if $c \leq a \leq b$, then $\frac{7}{2}c' \leq a' \leq b'$;*
- (ii) *if $\frac{7}{2}c \leq a \leq b$, then there exists $0 < \lambda_0 < 1$ such that $\frac{b}{a} \leq \lambda_0 \frac{b'}{a'}$.*

Proof. It is direct to check that $a \leq b$ implies $a' \leq b'$, and $c \leq b$ implies $\frac{a+b}{a+b+c} \geq \frac{1}{2}$. Hence

$$\frac{a'}{c'} \geq \frac{6a + 5c + \frac{1}{4}(a+c)}{c + \frac{1}{2}(a+c)} \geq \frac{7}{2}, \quad (5.2)$$

and the first inequality holds. To prove the second part, by (5.2), we have

$$\frac{b'}{a'} \geq \frac{6a + 8b + 5c + \frac{1}{4}(a+b)}{6a + 5c + \frac{1}{2}(a+c)} = \frac{20\left(\frac{c}{a}\right) + 25 + 33\left(\frac{b}{a}\right)}{22\left(\frac{c}{a}\right) + 26} \geq \frac{33\left(\frac{b}{a}\right)}{22\left(\frac{2}{7}\right) + 26} = \frac{231}{226} \cdot \frac{b}{a}.$$

The assertion follows by letting $\lambda_0 = \frac{226}{231}$. \square

Suppose each edge of G_n has resistance 1, then each edge in G'_n has resistance $(a_0, b_0, c_0) = \frac{1}{3}(1, 1, 1)$. Similar to the last example of the Viscek eyebolted cross, we can estimate the resulting resistance $\Phi^n(a_0, b_0, c_0) := (a_n, b_n, c_n)$ of G'_n on G'_0 .

Let $R_n(p_1, p_2)$, $R_n(p_2, p_3)$, $R_n(p_3, p_1)$ be the equivalent resulting resistance in the Δ -expression. Then using the $\Delta - Y$ transform, we have by (2.8),

$$\begin{aligned} R_n(p_1, p_2) &= a_n + b_n + \frac{a_n b_n}{c_n}, \\ R_n(p_2, p_3) &= b_n + c_n + \frac{b_n c_n}{a_n}, \\ R_n(p_3, p_1) &= c_n + a_n + \frac{c_n a_n}{b_n}. \end{aligned}$$

Proposition 5.3. *With the above expressions, we have*

$$R_n(p_1, p_2) \asymp R_n(p_2, p_3) \asymp \left(\frac{17}{2}\right)^n, \quad \text{and} \quad R_n(p_3, p_1) \asymp 7^n.$$

Proof. It follows from Lemma 5.2 that

$$\frac{c_n}{b_n} \leq \frac{a_n}{b_n} \leq \lambda_0^n. \quad (5.3)$$

As

$$\frac{b_n}{b_{n-1}} = 5 \cdot \frac{c_{n-1}}{b_{n-1}} + 6 \cdot \frac{a_{n-1}}{b_{n-1}} + 8 + \frac{\left(1 + \frac{a_{n-1}}{b_{n-1}}\right)\left(1 + \frac{c_{n-1}}{b_{n-1}}\right)}{2\left(1 + \frac{c_{n-1}}{b_{n-1}} + \frac{a_{n-1}}{b_{n-1}}\right)},$$

a direct estimation using (5.3) yields

$$\frac{17}{2} - \lambda_0^{n-1} \leq \frac{b_n}{b_{n-1}} \leq \frac{17}{2} + \frac{23}{2} \lambda_0^{n-1}. \quad (5.4)$$

which implies $b_n \asymp \left(\frac{17}{2}\right)^n$. On the other hand,

$$\begin{aligned} \frac{a_{n+1}}{c_{n+1}} - 11 &= \frac{5c_n + 6a_n + \frac{(a_n+b_n)(a_n+c_n)}{2(a_n+b_n+c_n)}}{c_n + \frac{(a_n+c_n)(b_n+c_n)}{2(a_n+b_n+c_n)}} - 11 \\ &= \frac{\left(2 + \frac{13a_n+c_n}{b_n}\right)\left(\frac{a_n}{c_n} - 11\right) + \frac{132a_n-12c_n}{b_n}}{2\left(1 + \frac{a_n+c_n}{b_n}\right) + \left(1 + \frac{c_n}{b_n}\right)\left(1 + \frac{a_n}{c_n}\right)}. \end{aligned} \quad (5.5)$$

By (5.3), and letting $\alpha_n = \left|\frac{a_n}{c_n} - 11\right|$, we see from the above,

$$\alpha_{n+1} \leq \delta \alpha_n + \gamma \lambda_0^n$$

for some $0 < \delta < 1$, $\gamma > 0$, and for n sufficiently large. An inductive argument shows that there exist n_0 and $0 < \lambda_1 < 1$ such that for $n > n_0$, we have $\left|\frac{a_n}{c_n} - 11\right| \leq \lambda_1 \left|\frac{a_n}{c_n} - 11\right|$. It follows that there is C_1 such that

$$\left|\frac{a_n}{c_n} - 11\right| \leq C_1 \lambda_1^n \quad \forall n \geq 0. \quad (5.6)$$

Now observing that

$$\frac{c_{n+1}}{c_n} = 1 + \frac{\left(1 + \frac{a_n}{c_n}\right)(b_n + c_n)}{2(a_n + b_n + c_n)} = 7 + \frac{\left(1 + \frac{c_n}{b_n}\right)\left(\frac{a_n}{c_n} - 11\right) - 12\frac{a_n}{b_n}}{2\left(1 + \frac{a_n+c_n}{b_n}\right)}, \quad (5.7)$$

By (5.3), (5.6) and (5.7), we conclude that $c_n \asymp 7^n$, and the same for a_n . The estimation of the $R_n(p_i, p_j)$ follows. \square

Theorem 5.4. *For the Besov spaces $B_{2,\infty}^\sigma$ defined on the Sierpinski sickle K , the critical exponents are*

$$\sigma^* = \frac{1}{2} \left(1 + \frac{\log 17}{\log 7}\right), \quad \sigma^\# = \frac{1}{2} \left(\frac{2 \log 17 - \log 2}{\log 7}\right)$$

Proof. By the resulting resistance, we see that the minimum of the following quadratic form taken over all functions u on V_n with fixed values $u(p_1), u(p_2), u(p_3)$ is given by

$$\begin{aligned} \min_u \left\{ \sum_{x,y \in V_\omega, |\omega|=n} |u(x) - u(y)|^2 \right\} &= \sum_{i \neq j} \frac{1}{R_n(p_i, p_j)} |u(p_i) - u(p_j)|^2 \\ &\asymp \left(\frac{2}{17}\right)^n |u(p_1) - u(p_2)|^2 + \left(\frac{2}{17}\right)^n |u(p_2) - u(p_3)|^2 + \left(\frac{1}{7}\right)^n |u(p_3) - u(p_1)|^2. \end{aligned} \quad (5.8)$$

Case for $\sigma^\#$: If we multiply $(\frac{17}{2})^n$ to both sides of (5.8), we have,

$$\min_u \left\{ \left(\frac{17}{2}\right)^n \sum_{x,y \in V_\omega, |\omega|=n} (u(x) - u(y))^2 \right\} \asymp (u(p_1) - u(p_2))^2 + (u(p_2) - u(p_3))^2 + \left(\frac{17}{14}\right)^n (u(p_3) - u(p_1))^2. \quad (5.9)$$

(It has a similar expression as in (4.3).) Let $\tau = \frac{1}{2} \left(\frac{2 \log 17 - \log 2}{\log 7} \right)$. Observe that $\rho = 1/7$ and the Hausdorff dimension of K is $\alpha = \log 17 / \log 7$, we have $\frac{17}{2} = \rho^{-(2\tau - \alpha)}$. It follows from the same argument as in the proof of Theorem 4.3 that $\sigma^\# \leq \tau$. The reverse inequality also holds by choosing u such that $u(p_1), u(p_3) \neq u(p_2)$, but $u(p_1) = u(p_3)$.

Case for σ^* : If we multiply 7^n on both sides of (5.8), we have

$$\min_u \left\{ 7^n \sum_{\substack{x,y \in V_\omega, \\ |\omega|=n}} |u(x) - u(y)|^2 \right\} \asymp \left(\frac{14}{17}\right)^n |u(p_1) - u(p_2)|^2 + \left(\frac{14}{17}\right)^n |u(p_2) - u(p_3)|^2 + |u(p_3) - u(p_1)|^2. \quad (5.10)$$

Then by observing that for $\sigma > \frac{1}{2} \left(1 + \frac{\log 17}{\log 7} \right)$, $B_{2,\infty}^\sigma$ is contained in the class of continuous functions u that are constants on each connected line segment parallel to $\overline{p_1 p_3}$ on K , and using the same proof as Theorem 4.3 again, we can show that $B_{2,\infty}^\sigma$ is not dense in $C(K)$. \square

In the following we will consider the density of the Besov spaces $B_{2,\infty}^\sigma$ in $C(K)$ and $L^2(K, \mu)$ at the critical exponents.

For $B_{2,\infty}^{\sigma^*}$, we first prove a lemma for a special energy functional on the V_n . Suppose we are given values $u(p_1) = x, u(p_2) = y$ and $u(p_3) = z$ with $x \neq z$. We define values of u on V_1 as indicated in Figure 7, with $q_0 = x$,

$$\begin{aligned} q_i &= q_{i-1} + \frac{(z-x)}{7}, \quad i = 1, 2, \dots, 6 \\ r_j &= q_{j-1} + \frac{(z-x)}{14}, \quad j = 1, 2, \dots, 5 \end{aligned}$$

$$s = \frac{x+z}{2}.$$

Then repeat this procedure on each sub-cell in V_1 , and extend u to V_2 , and so on, we obtain a continuous function on K , still denote by u . For $n \geq 0$, denote by $E_n(x, y, z) := \sum_{p, q \in V_\omega, |\omega|=n} (u \circ F_\omega(p) - u \circ F_\omega(q))^2$,

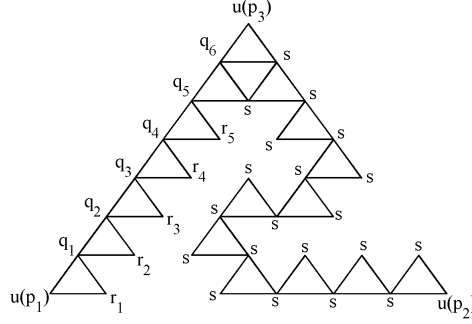


FIGURE 7. function u on V_1

Lemma 5.5. *There exists $C > 0$ such that for all $n \geq 0$,*

$$7^n E_n(x, y, z) \leq C((x-z)^2 + (x-y)^2 + (y-z)^2).$$

Remark. This is stronger than (5.10) since the energy minimizer function u is not fixed for different n there, and is not clear to converge, while here u is a fixed continuous function on K .

Proof. We observe that (see Figure 7)

$$E_n(x, y, z) = 5E_{n-1}(q_0, r_1, q_1) + E_{n-1}(q_6, s, z) + E_{n-1}(q_5, s, q_6) + E_{n-1}(s, y, s) \quad (5.11)$$

Denote the above E_{n-1} 's by I, II, III and IV , then make use of

$$E_k(x'-a, y'-a, z'-a) = E_k(x', y', z') \quad \text{and} \quad E_k(\eta x', \eta y', \eta z') = \eta^2 E_k(x', y', z'),$$

it is direct to check that

$$\begin{aligned} I &= \frac{1}{7^2} E_{n-1}\left(x, \frac{x+z}{2}, z\right), & II &= I + \frac{18}{49} \left(\frac{2}{17}\right)^{n-2} (x-z)^2 \\ III &= I + \frac{18}{49} \left(\frac{2}{17}\right)^{n-2} (x-z)^2, & IV &= 2 \left(\frac{2}{17}\right)^{n-1} \left(\frac{x+z}{2} - y\right)^2. \end{aligned}$$

Hence (5.11) is reduced to

$$E_n(x, y, z) = \frac{1}{7} E_{n-1}\left(x, \frac{x+z}{2}, z\right) + \frac{26}{49} \left(\frac{2}{17}\right)^{n-2} (x-z)^2 + 2 \left(\frac{2}{17}\right)^{n-1} \left(\frac{x+z}{2} - y\right)^2.$$

By induction, we have

$$\begin{aligned}
& E_n(x, y, z) \\
&= \frac{1}{7^{n-1}} E_1\left(x, \frac{x+z}{2}, z\right) + \frac{26}{49} (x-z)^2 \sum_{k=0}^{n-2} \left(\frac{1}{7}\right)^k \left(\frac{2}{17}\right)^{(n-2)-k} + 2 \left(\frac{2}{17}\right)^{n-1} \left(\frac{x+z}{2} - y\right)^2 \\
&\leq C_1 \frac{(x-z)^2}{7^n} + C_2 \frac{(x-z)^2}{7^n} + C_3 \left(\frac{2}{17}\right)^n \left(\frac{x+z}{2} - y\right)^2 \\
&\leq \frac{C}{7^n} \left((x-z)^2 + (x-y)^2 + (y-z)^2\right).
\end{aligned} \tag{5.12}$$

□

Proposition 5.6. *With the Sierpinski sickle K , we have*

(i) $B_{2,\infty}^{\sigma^*}$ is dense in $C(K)$.

(ii) For $\sigma^* < \sigma \leq \sigma^\#$, $B_{2,\infty}^\sigma$ is dense in $L^2(K, \mu)$, but not dense in $C(K)$.

Proof. (i) For any $f \in C(X)$ and for any $\varepsilon > 0$, let $m > 0$ be such that $|f(x) - f(y)| < \varepsilon$, $x, y \in V_n$. Denote $f_n = f|_{V_n}$, and apply the above lemma on each subcell $K_\omega \cap V_n$, then extend f_n to K continuously, we have $f_n \in B_{2,\infty}^{\sigma^*}$. As f on K_ω is bounded above and below by f on $K_\omega \cap V_n$, $\|f - f_n\| \leq \varepsilon$. This implies that $B_{2,\infty}^{\sigma^*}$ is dense in $C(K)$.

(ii) It suffices to prove that $B_{2,\infty}^{\sigma^\#}$ is dense in $L^2(K, \mu)$. Let L denote the line segment connecting p_1, p_3 , and let F_1, F_2, \dots, F_7 be the contractive maps along L . Let

$$L_m = \bigcup \{F_\omega(L) : |\omega| = m\}$$

be the set of line segments in K parallel to L . We can use the same argument in the last part of the proof of Proposition 4.3 to show that the set of continuous functions in $B_{2,\infty}^{\sigma^\#}$ and constant on each $F_\omega(L)$ is dense in $L^2(K, \mu)$. □

5.2. Dirichlet form.

We do not know if $B_{2,\infty}^{\sigma^*}$ can be domain of some local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$. But we can conclude that there does not exist such Dirichlet form with $B_{2,\infty}^{\sigma^*} = \mathcal{F}$ and $[u]_{B_{2,\infty}^{\sigma^*}}^2 \asymp \mathcal{E}[u]$ for all $u \in \mathcal{F}$. Indeed, if such \mathcal{E} exists, then we can construct a continuous function u on K such that $\mathcal{E}[u] = 0$, with $[u]_{B_{2,\infty}^{\sigma^*}} > 0$; the function is constant in each line segment in the $\overline{p_1 p_3}$ direction, we omit the detail.

On the other hand, we have the following conclusion.

Proposition 5.7. *The Sierpinski sickle admits a local regular Dirichlet form that satisfies the energy self-similar identity.*

Proof. We will determine three renormalizing factors τ_L, τ_R, τ_T on the cells of K as follows: let

$$\begin{aligned} \tau_1 &= \tau_2 = \cdots = \tau_5 = \tau_L \text{ on the left 5 sub-triangles } F_1(K), F_2(K), \dots, F_5(K); \\ \tau_6 &= \tau_7 = \tau_8 = \tau_T \text{ on the 3 top sub-triangles } F_6(K), F_7(K), F_8(K); \\ \tau_9 &= \tau_{10} = \cdots = \tau_{17} = \tau_R \text{ on the right 9 sub-triangles } F_9(K), F_{10}(K), \dots, F_{17}(K). \end{aligned}$$

Then similar to Lemma 5.1, we obtain the resulting resistance map:

$$\begin{aligned} \Phi_{\tau_L, \tau_R, \tau_T}(a, b, c) &= (a', b', c') \\ &= (\tau_L(5a + 5c) + \tau_T(a + \varphi_a), \tau_R(6a + 7b + 5c) + \tau_T(b + \varphi_b), \tau_T(c + \varphi_c)). \end{aligned}$$

where $\varphi_a = \frac{(a+b)(a+c)}{2(a+b+c)}$, and define φ_b, φ_c symmetrically. Let us take $a = b = kc$ with $k > 1$ and solve the equation

$$\Phi_{\tau_L, \tau_R, \tau_T}(a, b, c) = (a, b, c), \quad (5.13)$$

we obtain

$$\tau_L = \frac{k(k-1)}{5(k^2 + 6k + 3)}, \quad \tau_R = \frac{k^2 - 1}{(13k + 5)(k^2 + 6k + 3)}, \quad \tau_T = \frac{2(2k + 1)}{k^2 + 6k + 3}.$$

Let $E_0(u) = (u(p_1) - u(p_2))^2 + k(u(p_2) - u(p_3))^2 + k(u(p_3) - u(p_1))^2$ on V_0 , define

$$\mathcal{E}[u] = \lim_{n \rightarrow \infty} \sum_{|\omega|=n} \tau_\omega^{-1} E_0(u \circ F_\omega),$$

and let $\mathcal{F} = \{u \in C(K) : \mathcal{E}(u) < \infty\}$. Then $(\mathcal{E}, \mathcal{F})$ is a regular local Dirichlet form on $L^2(K, \mu)$, and satisfies the self-similar identity

$$\mathcal{E}[u] = \sum_{i=1}^{17} \tau_i^{-1} \mathcal{E}[u \circ F_i], \quad u \in \mathcal{F}.$$

□

Remark. Unlike the Vicsek eyebolted cross, we cannot get the other Dirichlet form on the Sierpinski sickle. Indeed, for any nonnegative initial value (a_0, b_0, c_0) with $a_0 + b_0 + c_0 = 1$ (we can assume this because $\Phi(\lambda(a_0, b_0, c_0)) = \lambda\Phi(a_0, b_0, c_0)$), let $(a_n, b_n, c_n) = \Phi^n(a_0, b_0, c_0)$. If $c_0 \leq a_0 \leq b_0$, then by Lemma 5.2, we see that there exists $0 < \lambda_0 < 1$ such that for all $n \geq 0$,

$$\frac{b_n}{a_n + c_n} \geq \frac{1}{2\lambda_0^n};$$

If $a_0 \leq c_0 \leq b_0$, then by a direct calculation, $b_1 \geq a_1 \geq c_1$ and reduces to the previous case. Finally if $b_0 \leq a_0$ (or $b_0 \leq c_0$), then $\frac{b_1}{a_1} \geq \frac{6a_0+5c_0+8b_0}{6a_0+5c_0+(a_0+b_0)/2} \geq \frac{12}{13}$, hence

$$\frac{b_2}{a_2} \geq \frac{6a_1 + 5c_1 + 8b_1}{6a_1 + 5c_1 + \frac{a_1+b_1}{2}} \geq 1.$$

Also we have $c_2 \leq a_2$ by similar calculation, and hence reduce back to the first case. We see that in all cases, $\frac{b_n}{a_n+c_n}$ goes to infinity very fast, that is

$$\frac{1}{a_n + b_n + c_n}(a_n, b_n, c_n) \rightarrow (0, 1, 0) \text{ uniformly as } n \rightarrow \infty.$$

Hence if we adopt the same method as in the second construction in Theorem 4.4, on the one hand, $\mathbf{y}_0 = (0, 1, 0)$ is not an interesting choice (as $\mathbf{y}_n = \Phi^{-n}(\mathbf{y}_0) = (\frac{2}{17})^n \mathbf{y}_0$ by (5.1)); on the other hand, for any initial value $\mathbf{y}_0 \neq (0, 1, 0)$, we can not expect to have a non-negative sequence $\{\mathbf{y}_n\}_n$ such that $\Phi(\mathbf{y}_n) = \mathbf{y}_{n-1}$ for all $n > 0$.

5.3. Other variances. The following discussion shows that we cannot reduce the maps in the Sierpinski sickle to obtain two critical exponents.

Proposition 5.8. *Consider the self-similar set K_1 generated by the IFS with 15 maps and contraction ratio $\rho = 1/7$ as shown in Fig 8. Then (as in Lemma 5.1), the relationship of the resistance of the cells on any two levels is given by :*

$$\Phi(a, b, c) = (6a + 5c + \varphi_a, 5a + 6b + 4c + \varphi_b, c + \varphi_c).$$

If we let $(a_n, b_n, c_n) = \Phi^{(n)}(1, 1, 1)$, then there exists $\lambda > 1$ such that

$$a_n \asymp b_n \asymp c_n \asymp \lambda^n. \quad (5.14)$$

Consequently, $B_{2,\infty}^{\sigma^*}$ is the domain of a Dirichlet form with uniform resistance ratio λ (similar to the Sierpinski gasket).

Proof. It is clear that $c_n \leq a_n \leq b_n$, and using this, it is not hard to show that $a_n \leq 34c_n$. Furthermore, we claim that $b_n \leq 60a_n$, then $a_n \asymp b_n \asymp c_n$.

To prove the claim, by using the fact that $a_n + 2b_n + c_n \geq \frac{4}{3}(a_n + b_n + c_n)$, we have

$$a_{n+1} + c_{n+1} = 6a_n + 6c_n + \frac{(a_n + c_n)(a_n + 2b_n + c_n)}{2(a_n + b_n + c_n)} \geq \frac{20}{3}(a_n + c_n). \quad (5.15)$$

On the other hand,

$$\begin{aligned} b_{n+1} &= 5a_n + 6b_n + 4c_n + \frac{(a_n + b_n)(b_n + c_n)}{2(a_n + b_n + c_n)} \\ &\leq 5a_n + \frac{13}{2}b_n + \frac{9}{2}c_n \leq \frac{13}{2}b_n + 5(a_n + c_n). \end{aligned} \quad (5.16)$$

Combining (5.15) and (5.16), and using induction, we obtain

$$b_n \leq 30(a_n + c_n) \leq 60a_n,$$

and the claim follows.

Note that $\Phi : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+^3$ satisfies $\Phi(\mathbf{x}) \leq \Phi(\mathbf{y})$ for any $\mathbf{x} \leq \mathbf{y}$ (coordinate-wise defined), and $\Phi(C\mathbf{x}) = C\Phi(\mathbf{x})$ for any $C > 0$. For $n, m \geq 1$, we have

$$\begin{aligned} (a_{m+n}, b_{m+n}, c_{m+n}) &= \Phi^{(n+m)}(1, 1, 1) = \Phi^{(m)}(a_n, b_n, c_n) = b_n \cdot \Phi^{(m)}\left(\frac{a_n}{b_n}, 1, \frac{c_n}{b_n}\right) \\ &\asymp b_n \cdot \Phi^{(m)}(1, 1, 1) = b_n \cdot (a_m, b_m, c_m). \end{aligned}$$

Then (5.14) follows by using a sub-additive argument. \square

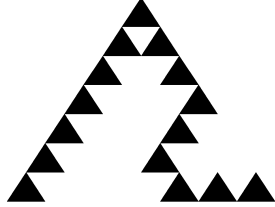


FIGURE 8. K_1

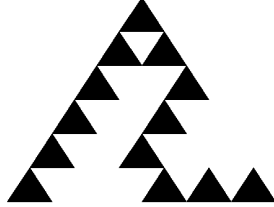


FIGURE 9. K_2

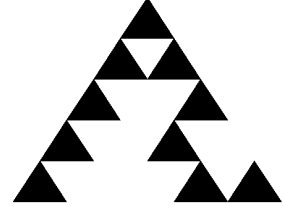


FIGURE 10. K_3

It is the same situation for $\rho = 1/6$ or $\rho = 1/5$ (Figure 9 and Figure 10). For K_2 , the resistances relationship can be written as (see (5.1))

$$\Phi(a, b, c) = (5a + 4c + \varphi_a, 4a + 5b + 4c + \varphi_b, c + \varphi_c).$$

and for K_3 ,

$$\Phi(a, b, c) = (4a + 3c + \varphi_a, 2a + 4b + 3c + \varphi_b, c + \varphi_c).$$

The same conclusion as Proposition 5.8 holds.

6. Remarks

In the study of Laplacian on fractals, the energy self-similar identity on the p.c.f. sets has been playing a central role in establishing the theory ([7, 12, 18]). In the two examples, we see that the renormalizing factors in the energy self-similar identity can be obtained from the compatibility of the resulting resistance $\Phi(\xi) = \xi$ (see (4.4) and (5.13)). This is a realization of the abstract proof of the existence of the Laplacian in literature through the Brouwer's fixed point theorem ([7, 12, 16]).

The alternative construction of the energy form on the Vicsek eyebolted cross is to use induction to find the resistance on the subcells ξ_n in V_n so that $\Phi^n(\xi_n) = \xi_0$ for a given ξ_0 . The construction seems to be quite flexible, but it also has limitation (as this method does not work for the Sierpinski sickle). It will be interesting to find out the validity of this method (for example the range of the above ξ_0) on the more general class of fractals, and to investigate the induced Laplacian, eigen-problems, heat kernels and the associated diffusion processes.

Besides the spaces $B_{2,\infty}^\sigma$, there is another important class of Besov spaces that is associated with the Dirichlet forms. Let

$$[u]_{B_{2,2}^\sigma}^2 := \int_K \int_K \frac{|u(x) - u(y)|^2}{|x - y|^{\alpha+2\sigma}} d\mu(y) d\mu(x), \quad (6.1)$$

and define $B_{2,2}^\sigma := \{u \in L^2(K, \mu) : \|u\|_{B_{2,2}^\sigma} < \infty\}$, with norm $\|u\|_{B_{2,2}^\sigma} := \|u\|_2 + [u]_{B_{2,2}^\sigma}$. This family of spaces is the domain of some non-local Dirichlet forms, and is associated with the class fractional Laplacians, and the stable jump processes [2]. It is not hard to see that the semi-norm $[u]_{B_{2,2}^\sigma}^2$ is equivalent to

$$\int_0^1 \frac{dr}{r} \cdot \frac{1}{r^{\alpha+2\sigma}} \int_K \int_{B(x,r)} (u(x) - u(y))^2 d\mu(y) d\mu(x). \quad (6.2)$$

Also, (6.2) can be expressed as

$$\sum_{n=0}^{\infty} \rho^{-n(\alpha+2\sigma)} \int_K \int_{B(x,\rho^n)} (u(x) - u(y))^2 d\mu(y) d\mu(x),$$

and similar to Theorem 3.4, we have the following discretization of $[u]_{B_{2,2}^\sigma}^2$.

Proposition 6.1. *Suppose the IFS $\{F_i\}_{i=1}^N$ is as in (2.1) and has the p.c.f. property. Then for $2\sigma > \alpha$,*

$$[u]_{B_{2,2}^\sigma}^2 \asymp \sum_{j=0}^{\infty} \rho^{-(2\sigma-\alpha)j} \sum_{x,y \in V_\omega, |\omega|=j} |u(x) - u(y)|^2. \quad (6.3)$$

The spaces $B_{2,2}^\sigma$ satisfy the following inclusion relation: for $0 < \varepsilon < \sigma$:

$$B_{2,2}^\sigma \subseteq B_{2,\infty}^\sigma \subseteq B_{2,2}^{\sigma-\varepsilon}.$$

Hence they share the same critical exponents as the class $B_{2,\infty}^\sigma$, $\sigma > 0$. In view of Proposition 5.6, we have

Corollary 6.2. *For the Sierpinski sickle, $B_{2,2}^{\sigma^*}$ is not dense in $C(K)$, and $B_{2,2}^{\sigma^\#}$ contains only constant functions. For the Vicsek eyebolted cross, $B_{2,2}^{\sigma^*}$ contains only constant functions.*

Proof. For any $u \in B_{2,2}^{\sigma^*}$, by discretizing $[u]_{B_{2,2}^{\sigma^*}}^2$ as (6.3), we have

$$\sum_{m=0}^{\infty} 7^m \sum_{x,y \in V_\omega; |\omega|=m} (u(x) - u(y))^2 \asymp [u]_{B_{2,2}^{\sigma^*}}^2 < \infty.$$

Then

$$\lim_{m \rightarrow \infty} 7^m \sum_{x,y \in V_\omega; |\omega|=m} (u(x) - u(y))^2 = 0. \quad (6.4)$$

From this, we claim that u is constant in the direction of $\overline{p_1 p_3}$. For if otherwise, then there must be some finite word τ such that $u \circ F_\tau(p_1) \neq u \circ F_\tau(p_3)$, then for any $n \geq 0$,

$$7^{|\tau|+n} \sum_{|\omega|=|\tau|+n} (u \circ F_\omega(p_1) - u \circ F_\omega(p_3))^2 \geq 7^{|\tau|} (u \circ F_\tau(p_1) - u \circ F_\tau(p_3))^2 > 0,$$

a contradiction to (6.4), and the claim follows. This implies that $B_{2,2}^{\sigma^*}$ is not dense in $C(K)$. We can also show that the rest statements are true by using similar arguments. \square

Finally we mention that in [13, 14], it is shown that on any self-similar set with the OSC, if we run certain reversible transient random walks on the “augmented tree” defined on the symbolic space by the neighbors of the cells, then they can induce the class of non-local Dirichlet forms as in (6.1). In the setup, the critical exponent for $\{B_{2,2}^\sigma\}_{\sigma>0}$ related to the random walk has been studied in detail. It remains one of the most interesting questions concerning the transition of $B_{2,2}^\sigma$ to $B_{2,\infty}^{\sigma^*}$, as $\sigma \rightarrow \sigma^*$.

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